

LOCALIZATION OF QUANTUM BIEQUIVARIANT \mathcal{D} -MODULES AND Q-W ALGEBRAS

A. SEVOSTYANOV

ABSTRACT. We present a biequivariant version of Kremnizer–Tanisaki localization theorem for quantum \mathcal{D} -modules. We also obtain an equivalence between a category of finitely generated equivariant modules over a quantum group and a category of finitely generated modules over a q-W algebra which can be regarded as an equivariant quantum group version of Skryabin equivalence. The biequivariant localization theorem for quantum \mathcal{D} -modules together with the equivariant quantum group version of Skryabin equivalence yield an equivalence between a certain category of quantum biequivariant \mathcal{D} -modules and a category of finitely generated modules over a q-W algebra.

1. INTRODUCTION

Let G be a complex simple connected simply connected algebraic group with Lie algebra \mathfrak{g} , B a Borel subgroup of G , \mathfrak{b} the Lie algebra of B . Denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} . Let λ be a weight of \mathfrak{g} , M_λ the Verma module over \mathfrak{g} with highest weight λ with respect to the system of positive roots of the pair $(\mathfrak{g}, \mathfrak{b})$. Denote by I_λ the annihilator of M_λ in $U(\mathfrak{g})$, and let $U(\mathfrak{g})^\lambda = U(\mathfrak{g})/I_\lambda$. Note that I_λ is generated by a maximal ideal of the center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ which is the kernel of a character $\chi_\lambda : Z(U(\mathfrak{g})) \rightarrow \mathbb{C}$. By the celebrated Beilinson–Bernstein theorem, if λ is regular dominant then the category of $U(\mathfrak{g})^\lambda$ -modules is equivalent to the category of modules over the sheaf $\mathcal{D}_{G/B}^\lambda$ of λ -twisted differential operators on the flag variety G/B which are quasi-coherent over the sheaf of regular functions $\mathbb{C}[G/B]$ on G/B . The functor providing the equivalence is simply the global section functor.

This result was generalized to the case of quantum groups in [1, 33]. The main observation used in [1] is that \mathcal{D}_λ can be regarded as a quantization of the λ -twisted cotangent bundle $T^*(G/B)_\lambda$ which is a symplectic leaf in the quotient $(T^*G)/B$ of the symplectic variety T^*G , equipped with the canonical symplectic structure of the cotangent bundle, by the Hamiltonian action induced by the B -action by right translations on G . Note that λ naturally gives rise to a character $\lambda : \mathfrak{b} \rightarrow \mathbb{C}$, and $T^*(G/B)_\lambda$ corresponds to the value $\lambda \in \mathfrak{b}^*$ of the moment map $\mu : T^*G \rightarrow \mathfrak{b}^*$ for the B -action, $T^*(G/B)_\lambda = \mu^{-1}(\lambda)/B$. Using this observation at the quantum level one can replace the category of $\mathcal{D}_{G/B}^\lambda$ -modules with a category \mathcal{D}_B^λ of modules over the sheaf of differential operators \mathcal{D}_G on G which are equivariant with respect to a left B -action. Objects of this category are \mathcal{D}_G -modules M equipped with the structure of B -modules in such a way that the action map $\mathcal{D}_G \otimes M \rightarrow M$ is a morphism of B -modules, where the action of B on \mathcal{D}_G is induced by the action on G by right translations, and the differential of the action of B on M coincides with the action of the Lie algebra \mathfrak{b} on the tensor product $M \otimes \mathbb{C}_\lambda$, where \mathfrak{b} acts on M via the natural embedding $\mathfrak{b} \rightarrow \mathcal{D}_G$, and \mathbb{C}_λ is the one-dimensional representation of \mathfrak{b} corresponding to the character λ . The Beilinson–Bernstein localization theorem for equivariant \mathcal{D}_G -modules was already formulated in [2] (see also [16] for some further details).

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Note that T^*G is naturally equipped with a G -action induced by the G -action by left translations on G . This action also preserves the canonical symplectic structure on T^*G and commutes with the right B -action. Hence it induces a Hamiltonian G -action on $(T^*G)/B$ and on all its symplectic leaves. In particular, the natural G -action on $T^*(G/B)_\lambda$ is Hamiltonian. One can restrict this action to various subgroups of G . Let N be such a subgroup with Lie algebra \mathfrak{n} equipped with a character $\chi : \mathfrak{n} \rightarrow \mathbb{C}$. Similarly to the case of B -equivariant \mathcal{D}_G -modules one can consider the category ${}^N_N \mathcal{D}_{G/B}^\lambda$ of N -equivariant $\mathcal{D}_{G/B}^\lambda$ -modules. By Beilinson–Bernstein localization theorem this category is equivalent to the category ${}^N_N U(\mathfrak{g})^\lambda$ -mod of equivariant (\mathfrak{g}, N) -modules on which the center $Z(U(\mathfrak{g}))$ acts by the character χ_λ . This category is defined similarly to the category \mathcal{D}_B^λ . Its objects are left \mathfrak{g} -modules V equipped with the structure of left N -modules in such a way that the action map $\mathfrak{g} \otimes V \rightarrow V$ is a morphism of N -modules, where the action of N on \mathfrak{g} is induced by the adjoint representation, and the differential of the action of N on V coincides with the action of the Lie algebra \mathfrak{n} on the tensor product $V \otimes \mathbb{C}_\chi$, where \mathfrak{n} acts on V via the natural embedding $\mathfrak{b} \rightarrow \mathfrak{g}$, and \mathbb{C}_χ is the one-dimensional representation of \mathfrak{n} corresponding to the character χ .

Now let $\mu_1 : T^*(G/B)_\lambda \rightarrow \mathfrak{n}^*$ be the moment map corresponding to the Hamiltonian group action of N on $T^*(G/B)_\lambda$, and ${}_\chi T^*(G/B)_\lambda = \mu_1^{-1}(\chi)/N$ the corresponding reduced Poisson manifold. Following the philosophy presented before in case of equivariant \mathcal{D}_G -modules one can expect that the category ${}^N_N \mathcal{D}_{G/B}^\lambda$ is equivalent to the category of \mathcal{D} -modules related to certain quantization of ${}_\chi T^*(G/B)_\lambda$, and the category ${}^N_N U(\mathfrak{g})^\lambda$ -mod is equivalent to the category of modules over an associative algebra ${}^N_N U(\mathfrak{g})^\lambda$ which is a quantization of ${}_\chi T^*(G/B)_\lambda$. Putting the two equivariance conditions together this would yield an equivalence between a category ${}^N_N \mathcal{D}_B^\lambda$ of \mathcal{D}_G -modules equipped with the two equivariance conditions with respect to actions of B and N and a category of ${}^N_N U(\mathfrak{g})^\lambda$ -modules.

Such equivalence was established, for instance, in case of modules over W -algebras in [9] when the subgroup N and its character χ are chosen in such a way that ${}^N_N U(\mathfrak{g})^\lambda$ is a quotient of a finitely generated W -algebra over a central ideal. In this paper we are going to obtain a similar categorial equivalence in case of q - W -algebras introduced in [32].

The definition of q - W -algebras is given in terms of quantum groups and we shall need an analogue of Beilinson–Bernstein localization for quantum groups. First of all there is a natural analogue of the algebra of differential operators on G for quantum groups called the Heisenberg double \mathcal{D}_q (see [26]). \mathcal{D}_q is a smash product of a quantum group $U_q(\mathfrak{g})$ and of the dual Hopf algebra generated by matrix elements of finite-dimensional representations of the quantum group. Similarly to the case of Lie algebras one can consider the category of \mathcal{D}_q -modules which are equivariant, in a sense similar to the Lie algebra case, with respect to a locally finite action of a quantum group analogue $U_q(\mathfrak{b}_+)$ of the universal enveloping algebra of a Borel subalgebra, $U_q(\mathfrak{b}_+)$ being equipped with a character λ as well. The main statement of [1] is that if λ is regular dominant the category of such \mathcal{D}_q -modules is equivalent to the category of $U_q(\mathfrak{g})^\lambda$ -modules, where $U_q(\mathfrak{g})^\lambda = U_q(\mathfrak{g})^{fin}/J_\lambda$, and $U_q(\mathfrak{g})^{fin}$ is the locally finite part with respect to the adjoint action of the Hopf algebra $U_q(\mathfrak{g})$ on itself, J_λ is the annihilator of the Verma module with highest weight λ in $U_q(\mathfrak{g})^{fin}$.

In Section 13 we give a quantum group analogue of the localization theorem for the category ${}^N_N \mathcal{D}_B^\lambda$. Our construction is a straightforward generalization of the classical result. Such an easy generalization is possible because the Heisenberg double is equipped with natural analogues of the G -actions on the algebra of differential operators on G induced by left and right translations on G .

This result can be applied in case of q - W -algebras if a quantum analogue of the group N and of its character are chosen in a proper way. Appropriate subalgebras $U_q^s(\mathfrak{m}_+)$ of $U_q(\mathfrak{g})$ with characters χ_q^s were defined in terms of certain new realizations $U_q^s(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ associated to Weyl group elements s of the Weyl group W of \mathfrak{g} . The definition of subalgebras $U_q^s(\mathfrak{m}_+)$ requires a deep study of the algebraic structure of $U_q(\mathfrak{g})$ presented in [32]. We recall the main results of [32] in Sections 4–11. However, the definition of the category of $U_q^s(\mathfrak{m}_+)$ -equivariant modules over

$U_q^s(\mathfrak{g})$ requires some further investigation presented in Sections 9–11. The problem is that a proper definition of this category formulated in Section 12 can only be given in terms of the locally finite part $U_q^s(\mathfrak{g})^{fin}$ of $U_q^s(\mathfrak{g})$, and the definition of the corresponding q-W-algebras associated to characters $\chi_q^s : U_q^s(\mathfrak{m}_+) \rightarrow \mathbb{C}$ given in Section 9 in terms of $U_q^s(\mathfrak{g})^{fin}$ also becomes more complicated comparing to the one suggested in [32]. The use of the locally finite part $U_q^s(\mathfrak{g})^{fin}$ is related to the fact that $U_q^s(\mathfrak{g})^{fin}$ is a deformation of the algebra of regular functions on the algebraic group G which follows from Proposition 9.3. Implicitly this result is also contained in [13].

The most difficult part of our construction is the proof of the equivalence between the category of finitely generated modules over $U_q^s(\mathfrak{g})^{fin}$ equivariant over $U_q^s(\mathfrak{m}_+)$ and the category of finitely generated modules over the corresponding q-W-algebra $W_q^s(G)$ which can be regarded as an equivariant version of Skryabin equivalence for quantum groups (see Appendix to [21]). We use the idea of the proof of a similar fact for W-algebras as it appears in [12]. However, technical difficulties in case of quantum groups become obscure. Our proof is presented in Section 12. It heavily relies on the behavior of all ingredients of the construction in the classical limit $q \rightarrow 1$. In particular, the key step is to use the cross-section theorem for the action of a unipotent algebraic subgroup $N \subset G$ on a subvariety of G obtained in [31]. Let $U(\mathfrak{m}_+)$ be $q = 1$ specialization of the $U_q^s(\mathfrak{m}_+)$. The cross-section theorem implies in particular that as a $U(\mathfrak{m}_+)$ -module the $q = 1$ specialization of any $U_q^s(\mathfrak{m}_+)$ -equivariant $U_q^s(\mathfrak{g})^{fin}$ -module V is isomorphic to the space of homomorphisms $\text{hom}_{\mathbb{C}}(U(\mathfrak{m}_+), V')$ of $U(\mathfrak{m}_+)$ into a vector space V' vanishing on some power of the natural augmentation ideal of $U(\mathfrak{m}_+)$.

The quantum group analogue of the localization theorem for the category ${}^X_N U(\mathfrak{g})^\lambda \text{--mod}$ easily gives an equivalence between a category of modules over \mathcal{D}_q equivariant with respect to a $U_q(\mathfrak{b}_+)$ -action and to a $U_q^s(\mathfrak{m}_+)$ -action and the category of $U_q^s(\mathfrak{m}_+)$ -equivariant modules over $U_q^s(\mathfrak{g})^{fin}$ with central character χ_λ . This equivalence together with the equivariant Skryabin equivalence for quantum groups yield an equivalence between a category of finitely generated modules over \mathcal{D}_q equivariant with respect to a $U_q(\mathfrak{b}_+)$ -action and to a $U_q^s(\mathfrak{m}_+)$ -action and the category of finitely generated modules over the quotient $W_q^s(G)_\lambda$ of the corresponding q-W-algebra $W_q^s(G)$ by a central ideal. This agrees with the general philosophy that $W_q^s(G)_\lambda$, or more generally $W_q^s(G)$, can be regarded as a quantization of the algebra of regular functions on a reduced Poisson manifold. In case of the algebra $W_q^s(G)$ the corresponding manifold is an algebraic group analogue of Slodowy slices associated to Weyl group element s (see Theorem 11.3). Such slices transversal to conjugacy classes in G were defined in [31].

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2. NOTATION

Fix the notation used throughout of the text. Let G be a connected finite-dimensional complex simple Lie group, \mathfrak{g} its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Let α_i , $i = 1, \dots, l$, $l = \text{rank}(\mathfrak{g})$ be a system of simple roots, $\Delta_+ = \{\beta_1, \dots, \beta_N\}$ the set of positive roots. Let H_1, \dots, H_l be the set of simple root generators of \mathfrak{h} .

Let a_{ij} be the corresponding Cartan matrix, and let d_1, \dots, d_l be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non-degenerate invariant symmetric bilinear form $(,)$ on \mathfrak{g} such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by α^\vee the element of \mathfrak{h} that corresponds to $\alpha \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on \mathfrak{h}^* is given by $(\alpha_i, \alpha_j) = b_{ij}$.

Let W be the Weyl group of the root system Δ . W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections s_1, \dots, s_l ,

$$s_i(h) = h - \alpha_i(h)H_i, \quad h \in \mathfrak{h}.$$

The action of W preserves the bilinear form (\cdot, \cdot) on \mathfrak{h} . We denote a representative of $w \in W$ in G by the same letter. For $w \in W, g \in G$ we write $w(g) = wgw^{-1}$. For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection.

Let \mathfrak{b}_+ be the positive Borel subalgebra and \mathfrak{b}_- the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nilradicals. Let $H = \exp \mathfrak{h}, N_+ = \exp \mathfrak{n}_+, N_- = \exp \mathfrak{n}_-, B_+ = HN_+, B_- = HN_-$ be the Cartan subgroup, the maximal unipotent subgroups and the Borel subgroups of G which correspond to the Lie subalgebras $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, \mathfrak{b}_+$ and \mathfrak{b}_- , respectively.

We identify \mathfrak{g} and its dual by means of the canonical invariant bilinear form. Then the coadjoint action of G on \mathfrak{g}^* is naturally identified with the adjoint one. We also identify $\mathfrak{n}_+^* \cong \mathfrak{n}_-, \mathfrak{b}_+^* \cong \mathfrak{b}_-$.

Let \mathfrak{g}_β be the root subspace corresponding to a root $\beta \in \Delta$, $\mathfrak{g}_\beta = \{x \in \mathfrak{g} | [h, x] = \beta(h)x \text{ for every } h \in \mathfrak{h}\}$. $\mathfrak{g}_\beta \subset \mathfrak{g}$ is a one-dimensional subspace. It is well known that for $\alpha \neq -\beta$ the root subspaces \mathfrak{g}_α and \mathfrak{g}_β are orthogonal with respect to the canonical invariant bilinear form. Moreover \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$ are non-degenerately paired by this form.

Root vectors $X_\alpha \in \mathfrak{g}_\alpha$ satisfy the following relations:

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})\alpha^\vee.$$

Note also that in this paper we denote by \mathbb{N} the set of nonnegative integer numbers, $\mathbb{N} = \{0, 1, \dots\}$.

3. QUANTUM GROUPS

In this section we recall some basic facts about quantum groups. We follow the notation of [7].

Let h be an indeterminate, $\mathbb{C}[[h]]$ the ring of formal power series in h . We shall consider $\mathbb{C}[[h]]$ -modules equipped with the so-called h -adic topology. For every such module V this topology is characterized by requiring that $\{h^n V \mid n \geq 0\}$ is a base of the neighborhoods of 0 in V , and that translations in V are continuous. It is easy to see that, for modules equipped with this topology, every $\mathbb{C}[[h]]$ -module map is automatically continuous.

A topological Hopf algebra over $\mathbb{C}[[h]]$ is a complete $\mathbb{C}[[h]]$ -module A equipped with a structure of $\mathbb{C}[[h]]$ -Hopf algebra (see [7], Definition 4.3.1), the algebraic tensor products entering the axioms of the Hopf algebra are replaced by their completions in the h -adic topology. We denote by $\mu, \iota, \Delta, \varepsilon, S$ the multiplication, the unit, the comultiplication, the counit and the antipode of A , respectively.

The standard quantum group $U_h(\mathfrak{g})$ associated to a complex finite-dimensional simple Lie algebra \mathfrak{g} is the algebra over $\mathbb{C}[[h]]$ topologically generated by elements $H_i, X_i^+, X_i^-, i = 1, \dots, l$, and with the following defining relations:

$$(3.1) \quad \begin{aligned} [H_i, H_j] &= 0, [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, \\ X_i^+ X_j^- - X_j^- X_i^+ &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \end{aligned}$$

$$\text{where } K_i = e^{d_i h H_i}, e^h = q, q_i = q^{d_i} = e^{d_i h},$$

and the quantum Serre relations:

$$(3.2) \quad \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (X_i^\pm)^{1-a_{ij}-r} X_j^\pm (X_i^\pm)^r = 0, \quad i \neq j,$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q! [n-m]_q!}, \quad [n]_q! = [n]_q \dots [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

$U_h(\mathfrak{g})$ is a topological Hopf algebra over $\mathbb{C}[[h]]$ with comultiplication defined by

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_h(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,$$

$$\Delta_h(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,$$

antipode defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i^+) = -X_i^+ K_i^{-1}, \quad S_h(X_i^-) = -K_i X_i^-,$$

and counit defined by

$$\varepsilon_h(H_i) = \varepsilon_h(X_i^\pm) = 0.$$

We shall also use the weight-type generators

$$Y_i = \sum_{j=1}^l d_i(a^{-1})_{ij} H_j,$$

and the elements $L_i = e^{hY_i}$. They commute with the root vectors X_i^\pm as follows:

$$(3.3) \quad L_i X_j^\pm L_i^{-1} = q_i^{\pm \delta_{ij}} X_j^\pm.$$

We also obviously have

$$(3.4) \quad L_i L_j = L_j L_i.$$

The Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the standard bialgebra structure on \mathfrak{g} , i.e. $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) = U(\mathfrak{g})$, $\Delta_h = \Delta \pmod{h}$, where Δ is the standard comultiplication on $U(\mathfrak{g})$, and

$$\frac{\Delta_h - \Delta_h^{opp}}{h} \pmod{h} = \delta,$$

where $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ is the standard cocycle on \mathfrak{g} , and $\Delta_h^{opp} = \sigma \Delta_h$, σ is the permutation in $U_h(\mathfrak{g})^{\otimes 2}$, $\sigma(x \otimes y) = y \otimes x$. Recall that

$$\delta(x) = (\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x) 2r_+, \quad r_+ \in \mathfrak{g} \otimes \mathfrak{g},$$

$$(3.5) \quad r_+ = \frac{1}{2} \sum_{i=1}^l Y_i \otimes H_i + \sum_{\beta \in \Delta_+} (X_\beta, X_{-\beta})^{-1} X_\beta \otimes X_{-\beta}.$$

Here $X_{\pm\beta} \in \mathfrak{g}_{\pm\beta}$ are root vectors of \mathfrak{g} . The element $r_+ \in \mathfrak{g} \otimes \mathfrak{g}$ is called a classical r -matrix.

$U_h(\mathfrak{g})$ is a quasitriangular Hopf algebra, i.e. there exists an invertible element $\mathcal{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, called a universal R -matrix, such that

$$(3.6) \quad \Delta_h^{opp}(a) = \mathcal{R} \Delta_h(a) \mathcal{R}^{-1} \text{ for all } a \in U_h(\mathfrak{g}),$$

and

$$(3.7) \quad \begin{aligned} (\Delta_h \otimes id) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{23}, \\ (id \otimes \Delta_h) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{12}, \end{aligned}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\sigma \otimes id) \mathcal{R}_{23}$.

From (3.6) and (3.7) it follows that \mathcal{R} satisfies the quantum Yang–Baxter equation:

$$(3.8) \quad \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}.$$

For every quasitriangular Hopf algebra we also have (see Proposition 4.2.7 in [7]):

$$(S \otimes id) \mathcal{R} = (id \otimes S^{-1}) \mathcal{R} = \mathcal{R}^{-1},$$

and

$$(3.9) \quad (S \otimes S)\mathcal{R} = \mathcal{R}.$$

We shall explicitly describe the element \mathcal{R} . First following [7] we recall the construction of root vectors of $U_h(\mathfrak{g})$ in terms of a braid group action on $U_h(\mathfrak{g})$. Let m_{ij} , $i \neq j$ be equal to 2, 3, 4, 6 if $a_{ij}a_{ji}$ is equal to 0, 1, 2, 3. The braid group $\mathcal{B}_{\mathfrak{g}}$ associated to \mathfrak{g} has generators T_i , $i = 1, \dots, l$, and defining relations

$$T_i T_j T_i T_j \dots = T_j T_i T_j T_i \dots$$

for all $i \neq j$, where there are m_{ij} T 's on each side of the equation.

There is an action of the braid group $\mathcal{B}_{\mathfrak{g}}$ by algebra automorphisms of $U_h(\mathfrak{g})$ defined on the standard generators as follows:

$$T_i(X_i^+) = -X_i^- e^{hd_i H_i}, \quad T_i(X_i^-) = -e^{-hd_i H_i} X_i^+, \quad T_i(H_j) = H_j - a_{ji} H_i,$$

$$T_i(X_j^+) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} (X_i^+)^{(-a_{ij}-r)} X_j^+ (X_i^+)^{(r)}, \quad i \neq j,$$

$$T_i(X_j^-) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^r (X_i^-)^{(r)} X_j^- (X_i^-)^{(-a_{ij}-r)}, \quad i \neq j,$$

where

$$(X_i^+)^{(r)} = \frac{(X_i^+)^r}{[r]_{q_i}!}, \quad (X_i^-)^{(r)} = \frac{(X_i^-)^r}{[r]_{q_i}!}, \quad r \geq 0, \quad i = 1, \dots, l.$$

Recall that an ordering of a set of positive roots Δ_+ is called normal if all simple roots are written in an arbitrary order, and for any three roots α , β , γ such that $\gamma = \alpha + \beta$ we have either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$.

Any two normal orderings in Δ_+ can be reduced to each other by the so-called elementary transpositions (see [36], Theorem 1). The elementary transpositions for rank 2 root systems are inversions of the following normal orderings (or the inverse normal orderings):

$$(3.10) \quad \begin{array}{ll} \alpha, \beta & A_1 + A_1 \\ \alpha, \alpha + \beta, \beta & A_2 \\ \alpha, \alpha + \beta, \alpha + 2\beta, \beta & B_2 \\ \alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta & G_2 \end{array}$$

where it is assumed that $(\alpha, \alpha) \geq (\beta, \beta)$. Moreover, any normal ordering in a rank 2 root system is one of orderings (3.10) or one of the inverse orderings.

In general an elementary inversion of a normal ordering in a set of positive roots Δ_+ is the inversion of an ordered segment of form (3.10) (or of a segment with the inverse ordering) in the ordered set Δ_+ , where $\alpha - \beta \notin \Delta$.

For any reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of \mathfrak{g} the set

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_D = s_{i_1} \dots s_{i_{D-1}} \alpha_{i_D}$$

is a normal ordering in Δ_+ , and there is one-to-one correspondence between normal orderings of Δ_+ and reduced decompositions of w_0 (see [37]).

Now fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_D}$ of the longest element w_0 of the Weyl group W of \mathfrak{g} and define the corresponding root vectors in $U_h(\mathfrak{g})$ by

$$(3.11) \quad X_{\beta_k}^{\pm} = T_{i_1} \dots T_{i_{k-1}} X_{i_k}^{\pm}.$$

Proposition 3.1. *For $\beta = \sum_{i=1}^l m_i \alpha_i$, $m_i \in \mathbb{N}$ X_{β}^{\pm} is a polynomial in the noncommutative variables X_i^{\pm} homogeneous in each X_i^{\pm} of degree m_i .*

The root vectors X_{β}^{\pm} satisfy the following relations:

$$(3.12) \quad X_{\alpha}^{+} X_{\beta}^{+} - q^{(\alpha, \beta)} X_{\beta}^{+} X_{\alpha}^{+} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C(k_1, \dots, k_n) (X_{\delta_1}^{+})^{(k_1)} (X_{\delta_2}^{+})^{(k_2)} \dots (X_{\delta_n}^{+})^{(k_n)},$$

where for $\alpha \in \Delta_+$ we put $(X_{\alpha}^{\pm})^{(k)} = \frac{(X_{\alpha}^{\pm})^k}{[k]_{q_{\alpha}}!}$, $k \geq 0$, $q_{\alpha} = q^{d_i}$ if the positive root α is Weyl group conjugate to the simple root α_i , $C(k_1, \dots, k_n) \in \mathbb{C}[q, q^{-1}]$. They also commute with elements of the subalgebra $U_h(\mathfrak{h})$ as follows:

$$(3.13) \quad [H_i, X_{\beta}^{\pm}] = \pm \beta(H_i) X_{\beta}^{\pm}, \quad i = 1, \dots, l.$$

Note that by construction

$$X_{\beta}^{+} \pmod{h} = X_{\beta} \in \mathfrak{g}_{\beta},$$

$$X_{\beta}^{-} \pmod{h} = X_{-\beta} \in \mathfrak{g}_{-\beta}$$

are root vectors of \mathfrak{g} .

Let $U_h(\mathfrak{n}_+)$, $U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$ be the $\mathbb{C}[[h]]$ -subalgebras of $U_h(\mathfrak{g})$ topologically generated by the X_i^{+} , by the X_i^{-} and by the H_i , respectively.

Now using the root vectors X_{β}^{\pm} we can construct a topological basis of $U_h(\mathfrak{g})$. Define for $\mathbf{r} = (r_1, \dots, r_D) \in \mathbb{N}^D$,

$$(X^{+})^{(\mathbf{r})} = (X_{\beta_1}^{+})^{(r_1)} \dots (X_{\beta_D}^{+})^{(r_D)},$$

$$(X^{-})^{(\mathbf{r})} = (X_{\beta_D}^{-})^{(r_D)} \dots (X_{\beta_1}^{-})^{(r_1)},$$

and for $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{N}^l$,

$$H^{\mathbf{s}} = H_1^{s_1} \dots H_l^{s_l}.$$

Proposition 3.2. ([17], **Proposition 3.3**) *The elements $(X^{+})^{(\mathbf{r})}$, $(X^{-})^{(\mathbf{t})}$ and $H^{\mathbf{s}}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$, $\mathbf{s} \in \mathbb{N}^l$, form topological bases of $U_h(\mathfrak{n}_+)$, $U_h(\mathfrak{n}_-)$ and $U_h(\mathfrak{h})$, respectively, and the products $(X^{+})^{(\mathbf{r})} H^{\mathbf{s}} (X^{-})^{(\mathbf{t})}$ form a topological basis of $U_h(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules:*

$$U_h(\mathfrak{n}_-) \otimes U_h(\mathfrak{h}) \otimes U_h(\mathfrak{n}_+) \rightarrow U_h(\mathfrak{g}).$$

An explicit expression for \mathcal{R} may be written by making use of the q -exponential

$$\exp_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} \frac{x^k}{[k]_q!}$$

in terms of which the element \mathcal{R} takes the form:

$$(3.14) \quad \mathcal{R} = \exp \left[h \sum_{i=1}^l (Y_i \otimes H_i) \right] \prod_{\beta} \exp_{q_{\beta}} [(1 - q_{\beta}^{-2}) X_{\beta}^{+} \otimes X_{\beta}^{-}],$$

where the product is over all the positive roots of \mathfrak{g} , and the order of the terms is such that the α -term appears to the left of the β -term if $\alpha > \beta$ with respect to the normal ordering of Δ_+ .

Remark 3.1. The r -matrix $r_+ = \frac{1}{2}h^{-1}(\mathcal{R} - 1 \otimes 1) \pmod{h}$, which is the classical limit of \mathcal{R} , coincides with the classical r -matrix (3.5).

One can calculate the action of the comultiplication on the root vectors $X_{\beta_k}^{\pm}$ in terms of the universal \mathcal{R} -matrix. For instance for $\Delta_h(X_{\beta_k}^+)$ one has

$$(3.15) \quad \Delta_h(X_{\beta_k}^+) = \tilde{\mathcal{R}}_{<\beta_k}^{-1} (X_{\beta_k}^+ \otimes e^{h\beta_k^\vee} + 1 \otimes X_{\beta_k}^+) \tilde{\mathcal{R}}_{<\beta_k},$$

where

$$\tilde{\mathcal{R}}_{<\beta_k} = \tilde{\mathcal{R}}_{\beta_{k-1}} \dots \tilde{\mathcal{R}}_{\beta_1}, \quad \tilde{\mathcal{R}}_{\beta_r} = \exp_{q_{\beta_r}}[(1 - q_{\beta_r}^{-2})X_{\beta_r}^+ \otimes X_{\beta_r}^-].$$

4. REALIZATIONS OF QUANTUM GROUPS ASSOCIATED TO WEYL GROUP ELEMENTS

The subalgebras of $U_h(\mathfrak{g})$ which possess nontrivial characters are defined in terms of the new realizations $U_h^s(\mathfrak{g})$ of $U_h(\mathfrak{g})$ associated to Weyl group elements, and we start by defining these new realizations.

Let s be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, and \mathfrak{h}' the orthogonal complement, with respect to the Killing form, to the subspace of \mathfrak{h} fixed by the natural action of s on \mathfrak{h} . The restriction of the natural action of s on \mathfrak{h}^* to the subspace \mathfrak{h}'^* has no fixed points. Therefore one can define the Cayley transform $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ of the restriction of s to \mathfrak{h}'^* , where $P_{\mathfrak{h}'^*}$ is the orthogonal projection operator onto \mathfrak{h}'^* in \mathfrak{h}^* , with respect to the Killing form.

Now we suggest a new realization of the quantum group $U_h(\mathfrak{g})$ associated to $s \in W$. Let $U_h^s(\mathfrak{g})$ be the associative algebra over $\mathbb{C}[[h]]$ topologically generated by elements e_i, f_i, H_i , $i = 1, \dots, l$ subject to the relations:

$$(4.1) \quad \begin{aligned} [H_i, H_j] &= 0, \quad [H_i, e_j] = a_{ij}e_j, \quad [H_i, f_j] = -a_{ij}f_j, \\ e_i f_j - q^{c_{ij}} f_j e_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad c_{ij} = \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \alpha_i, \alpha_j \right), \\ K_i &= e^{d_i h H_i}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r &= 0, \quad i \neq j, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (f_i)^{1-a_{ij}-r} f_j (f_i)^r &= 0, \quad i \neq j. \end{aligned}$$

Theorem 4.1. ([32], Theorem 4.1) For every solution $n_{ij} \in \mathbb{C}$, $i, j = 1, \dots, l$ of equations

$$(4.2) \quad d_j n_{ij} - d_i n_{ji} = c_{ij}$$

there exists an algebra isomorphism $\psi_{\{n\}} : U_h^s(\mathfrak{g}) \rightarrow U_h(\mathfrak{g})$ defined by the formulas:

$$\psi_{\{n\}}(e_i) = X_i^+ \prod_{p=1}^l L_p^{n_{ip}},$$

$$\psi_{\{n\}}(f_i) = \prod_{p=1}^l L_p^{-n_{ip}} X_i^-,$$

$$\psi_{\{n\}}(H_i) = H_i.$$

Remark 4.2. The general solution of equation (4.2) is given by

$$(4.3) \quad n_{ij} = \frac{1}{2d_j} (c_{ij} + s_{ij}),$$

where $s_{ij} = s_{ji}$.

We call the algebra $U_h^s(\mathfrak{g})$ the realization of the quantum group $U_h(\mathfrak{g})$ corresponding to the element $s \in W$.

Remark 4.3. Let n_{ij} be a solution of the homogeneous system that corresponds to (4.2),

$$d_i n_{ji} - d_j n_{ij} = 0.$$

Then the map defined by

$$(4.4) \quad \begin{aligned} X_i^+ &\mapsto X_i^+ \prod_{p=1}^l L_p^{n_{ip}}, \\ X_i^- &\mapsto \prod_{p=1}^l L_p^{-n_{ip}} X_i^-, \\ H_i &\mapsto H_i \end{aligned}$$

is an automorphism of $U_h(\mathfrak{g})$. Therefore for given element $s \in W$ the isomorphism $\psi_{\{n\}}$ is defined uniquely up to automorphisms (4.4) of $U_h(\mathfrak{g})$.

Now we shall study the algebraic structure of $U_h^s(\mathfrak{g})$. Denote by $U_h^s(\mathfrak{n}_{\pm})$ the subalgebra in $U_h^s(\mathfrak{g})$ generated by $e_i (f_i), i = 1, \dots, l$. Let $U_h^s(\mathfrak{h})$ be the subalgebra in $U_h^s(\mathfrak{g})$ generated by $H_i, i = 1, \dots, l$.

We shall construct a Poincaré–Birkhoff–Witt basis for $U_h^s(\mathfrak{g})$. It is convenient to introduce an operator $K \in \text{End } \mathfrak{h}$ such that

$$(4.5) \quad KH_i = \sum_{j=1}^l \frac{n_{ij}}{d_i} Y_j.$$

In particular, we have

$$\frac{n_{ji}}{d_j} = (KH_j, H_i).$$

Equation (4.2) is equivalent to the following equation for the operator K :

$$K - K^* = \frac{1+s}{1-s} P_{\mathfrak{h}'*}.$$

Proposition 4.2. ([32], **Proposition 4.2**) (i) For any solution of equation (4.2) and any normal ordering of the root system Δ_+ the elements $e_{\beta} = \psi_{\{n\}}^{-1}(X_{\beta}^+ e^{hK\beta^{\vee}})$ and $f_{\beta} = \psi_{\{n\}}^{-1}(e^{-hK\beta^{\vee}} X_{\beta}^-)$, $\beta \in \Delta_+$ lie in the subalgebras $U_h^s(\mathfrak{n}_+)$ and $U_h^s(\mathfrak{n}_-)$, respectively. For $\alpha \in \Delta_+$ we put $(e_{\alpha})^{(k)} = \frac{(e_{\alpha})^k}{[k]_{q\alpha}!}$, $(f_{\alpha})^{(k)} = \frac{(f_{\alpha})^k}{[k]_{q\alpha}!}$, $k \geq 0$. The elements e_{β} , $\beta \in \Delta_+$ satisfy the following commutation relations

$$(4.6) \quad e_{\alpha} e_{\beta} - q^{(\alpha, \beta) + (\frac{1+s}{1-s} P_{\mathfrak{h}'*} \alpha, \beta)} e_{\beta} e_{\alpha} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C'(k_1, \dots, k_n) (e_{\delta_1})^{(k_1)} (e_{\delta_2})^{(k_2)} \dots (e_{\delta_n})^{(k_n)},$$

where $C'(k_1, \dots, k_n) \in \mathbb{C}[q, q^{-1}, q^{n_{ij}d_j}, q^{-n_{ij}d_j}]_{i,j=1,\dots,l}$.

(ii) Moreover, the elements $(e)^{(\mathbf{r})} = (e_{\beta_1})^{(r_1)} \dots (e_{\beta_D})^{(r_D)}$, $(f)^{(\mathbf{t})} = (f_{\beta_D})^{(t_D)} \dots (f_{\beta_1})^{(t_1)}$ and $H^{\mathbf{s}} = H_1^{s_1} \dots H_l^{s_l}$ for $\mathbf{r}, \mathbf{t}, \mathbf{s} \in \mathbb{N}^l$ form topological bases of $U_h^s(\mathfrak{n}_+)$, $U_h^s(\mathfrak{n}_-)$ and $U_h^s(\mathfrak{h})$, and the products $(f)^{(\mathbf{t})} H^{\mathbf{s}} (e)^{(\mathbf{r})}$ form a topological basis of $U_h^s(\mathfrak{g})$. In particular, multiplication defines an isomorphism of $\mathbb{C}[[h]]$ modules

$$U_h^s(\mathfrak{n}_-) \otimes U_h^s(\mathfrak{h}) \otimes U_h^s(\mathfrak{n}_+) \rightarrow U_h^s(\mathfrak{g}).$$

The realizations $U_h^s(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ are connected with quantizations of some nonstandard bialgebra structures on \mathfrak{g} . At the quantum level changing bialgebra structure corresponds to the so-called Drinfeld twist. We shall consider a particular class of such twists described in the following proposition.

Proposition 4.3. ([7], **Proposition 4.2.13**) *Let $(A, \mu, \iota, \Delta, \varepsilon, S)$ be a Hopf algebra over a commutative ring. Let \mathcal{F} be an invertible element of $A \otimes A$ such that*

$$(4.7) \quad \begin{aligned} \mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) &= \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F}), \\ (\varepsilon \otimes id)(\mathcal{F}) &= (id \otimes \varepsilon)(\mathcal{F}) = 1. \end{aligned}$$

Then, $v = \mu(id \otimes S)(\mathcal{F})$ is an invertible element of A with

$$v^{-1} = \mu(S \otimes id)(\mathcal{F}^{-1}).$$

Moreover, if we define $\Delta^{\mathcal{F}} : A \rightarrow A \otimes A$ and $S^{\mathcal{F}} : A \rightarrow A$ by

$$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(a) = vS(a)v^{-1},$$

then $(A, \mu, \iota, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a Hopf algebra denoted by $A^{\mathcal{F}}$ and called the twist of A by \mathcal{F} .

Corollary 4.4. ([7], **Corollary 4.2.15**) *Suppose that A and \mathcal{F} are as in Proposition 4.3, but assume in addition that A is quasitriangular with universal R -matrix \mathcal{R} . Then $A^{\mathcal{F}}$ is quasitriangular with universal R -matrix*

$$(4.8) \quad \mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1},$$

where $\mathcal{F}_{21} = \sigma\mathcal{F}$.

Fix an element $s \in W$. Consider the twist of the Hopf algebra $U_h(\mathfrak{g})$ by the element

$$(4.9) \quad \mathcal{F} = \exp(-h \sum_{i,j=1}^l \frac{n_{ji}}{d_j} Y_i \otimes Y_j) \in U_h(\mathfrak{h}) \otimes U_h(\mathfrak{h}),$$

where n_{ij} is a solution of the corresponding equation (4.2).

This element satisfies conditions (4.7), and so $U_h(\mathfrak{g})^{\mathcal{F}}$ is a quasitriangular Hopf algebra with the universal R -matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$, where \mathcal{R} is given by (3.14). We shall explicitly calculate the element $\mathcal{R}^{\mathcal{F}}$. Substituting (3.14) and (4.9) into (4.8) and using (3.13) we obtain

$$\begin{aligned} \mathcal{R}^{\mathcal{F}} &= \exp \left[h \left(\sum_{i=1}^l (Y_i \otimes H_i) + \sum_{i,j=1}^l \left(-\frac{n_{ij}}{d_i} + \frac{n_{ji}}{d_j} \right) Y_i \otimes Y_j \right) \right] \times \\ &\prod_{\beta} \exp_{q_{\beta}^{-1}} \left[(1 - q_{\beta}^{-2}) X_{\beta}^{+} e^{hK\beta^{\vee}} \otimes e^{-hK^{*}\beta^{\vee}} X_{\beta}^{-} \right], \end{aligned}$$

where K is defined by (4.5).

Equip $U_h^s(\mathfrak{g})$ with the comultiplication given by : $\Delta_s(x) = (\psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1}) \Delta_h^{\mathcal{F}}(\psi_{\{n\}}(x))$. Then $U_h^s(\mathfrak{g})$ becomes a quasitriangular Hopf algebra with the universal R -matrix $\mathcal{R}^s = \psi_{\{n\}}^{-1} \otimes \psi_{\{n\}}^{-1} \mathcal{R}^{\mathcal{F}}$. Using equation (4.2) this R -matrix may be written as follows

$$(4.10) \quad \begin{aligned} \mathcal{R}^s &= \exp \left[h \left(\sum_{i=1}^l (Y_i \otimes H_i) + \sum_{i=1}^l \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i \right) \right] \times \\ &\prod_{\beta} \exp_{q_{\beta}} \left[(1 - q_{\beta}^{-2}) e_{\beta} \otimes e^{h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^{\vee}} f_{\beta} \right], \end{aligned}$$

where $P_{\mathfrak{h}'}$ is the orthogonal projection operator onto \mathfrak{h}' in \mathfrak{h} with respect to the Killing form.

The element \mathcal{R}^s may be also represented in the form

$$(4.11) \quad \begin{aligned} \mathcal{R}^s &= \prod_{\beta} \exp_{q_{\beta}} \left[(1 - q_{\beta}^{-2}) e_{\beta} e^{-h \left(\frac{1+s}{1-s} P_{\mathfrak{h}'} + 1 \right) \beta^{\vee}} \otimes e^{h \beta^{\vee}} f_{\beta} \right] \times \\ &\exp \left[h \left(\sum_{i=1}^l (Y_i \otimes H_i) + \sum_{i=1}^l \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i \right) \right]. \end{aligned}$$

The comultiplication Δ_s is given on generators by

$$\begin{aligned}\Delta_s(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\ \Delta_s(e_i) &= e_i \otimes e^{hd_i(\frac{2}{1-s}P_{\mathfrak{h}'}+P_{\mathfrak{h}'^\perp})H_i} + 1 \otimes e_i, \\ \Delta_s(f_i) &= f_i \otimes e^{-hd_i\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i} + e^{-hd_iH_i} \otimes f_i,\end{aligned}$$

where $P_{\mathfrak{h}'^\perp}$ is the orthogonal projection operator onto the orthogonal complement \mathfrak{h}'^\perp to \mathfrak{h}' in \mathfrak{h} with respect to the Killing form.

Finally, the new antipode $S_s(x) = \psi_{\{n\}}^{-1} S_h^{\mathcal{F}}(\psi_{\{n\}}(x))$ is given by

$$S_s(e_i) = -e_i e^{-hd_i(\frac{2}{1-s}P_{\mathfrak{h}'}+P_{\mathfrak{h}'^\perp})H_i}, \quad S_s(f_i) = -e^{hd_iH_i} f_i e^{hd_i\frac{1+s}{1-s}P_{\mathfrak{h}'}H_i}, \quad S_s(H_i) = -H_i.$$

Note that the Hopf algebra $U_h^s(\mathfrak{g})$ is a quantization of the bialgebra structure on \mathfrak{g} defined by the cocycle

$$(4.12) \quad \delta(x) = (\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x) 2r_+^s, \quad r_+^s \in \mathfrak{g} \otimes \mathfrak{g},$$

where $r_+^s = r_+ + \frac{1}{2} \sum_{i=1}^l \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i$, and r_+ is given by (3.5).

Using formula (3.15) and Proposition 4.3 one can also easily find that

$$(4.13) \quad \Delta_s(e_{\beta_k}) = (\tilde{\mathcal{R}}_{<\beta_k}^s)^{-1} (e_{\beta_k} \otimes e^{h\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} + 1 \otimes e_{\beta_k}) \tilde{\mathcal{R}}_{<\beta_k}^s,$$

where

$$\tilde{\mathcal{R}}_{<\beta_k}^s = \tilde{\mathcal{R}}_{\beta_{k-1}}^s \dots \tilde{\mathcal{R}}_{\beta_1}^s, \quad \tilde{\mathcal{R}}_{\beta_r}^s = \exp_{q_{\beta_r}} [(1 - q_{\beta_r}^{-2}) e_{\beta_r} \otimes e^{h\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} f_{\beta_r}],$$

and

$$(\tilde{\mathcal{R}}_{<\beta_k}^s)^{-1} = (\tilde{\mathcal{R}}_{\beta_1}^s)^{-1} \dots (\tilde{\mathcal{R}}_{\beta_{k-1}}^s)^{-1}, \quad (\tilde{\mathcal{R}}_{\beta_r}^s)^{-1} = \exp_{q_{\beta_r}^{-1}} [(1 - q_{\beta_r}^2) e_{\beta_r} \otimes e^{h\frac{1+s}{1-s}P_{\mathfrak{h}'}\beta^\vee} f_{\beta_r}].$$

5. NILPOTENT SUBALGEBRAS AND QUANTUM GROUPS

First we recall the definition of certain normal orderings of root systems associated to Weyl group elements introduced in [32]. The definition of subalgebras of $U_h(\mathfrak{g})$ having nontrivial characters is given in terms of root vectors associated to such normal orderings.

Let s , as in the previous section, be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{h}_{\mathbb{R}}$ the real form of \mathfrak{h} , the real linear span of simple coroots in \mathfrak{h} . The set of roots Δ is a subset of the dual space $\mathfrak{h}_{\mathbb{R}}^*$.

The Weyl group element s naturally acts on $\mathfrak{h}_{\mathbb{R}}$ as an orthogonal transformation with respect to the scalar product induced by the Killing form of \mathfrak{g} . Using the spectral theory of orthogonal transformations we can decompose $\mathfrak{h}_{\mathbb{R}}$ into a direct orthogonal sum of s -invariant subspaces,

$$(5.1) \quad \mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^K \mathfrak{h}_i,$$

where we assume that \mathfrak{h}_0 is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s , and each of the other subspaces $\mathfrak{h}_i \subset \mathfrak{h}_{\mathbb{R}}$, $i = 1, \dots, K$, is either two-dimensional or one-dimensional and the Weyl group element s acts on it as rotation with angle θ_i , $0 < \theta_i \leq \pi$ or as reflection with respect to the origin (which also can be regarded as rotation with angle π). Note that since s has finite order $\theta_i = \frac{2\pi}{m_i}$, $m_i \in \mathbb{N}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \dots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s -invariant subspace \mathfrak{h}_i with respect to the natural pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$.

Now we consider certain s -invariant subsets of roots $\overline{\Delta}_i$, $i = 0, \dots, K$, defined as follows

$$(5.2) \quad \overline{\Delta}_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i, h_i(\alpha) \neq 0\},$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets $\overline{\Delta}_i$ are empty, and that the definition of these subsets depends on the order of terms in direct sum (5.1).

Now consider the nonempty s -invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0, \dots, M$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$. According to this definition $\overline{\Delta}_{i_0} = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s . Observe also that the root system Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$,

$$\Delta = \bigcup_{k=0}^M \overline{\Delta}_{i_k}.$$

Now assume that

$$(5.3) \quad |h_{i_k}(\alpha)| > \left| \sum_{l \leq j < k} h_{i_j}(\alpha) \right|, \text{ for any } \alpha \in \overline{\Delta}_{i_k}, \quad k = 0, \dots, M, \quad l < k.$$

Condition (5.3) can be always fulfilled by suitable rescalings of the elements h_{i_k} .

Consider the element

$$(5.4) \quad \bar{h} = \sum_{k=0}^M h_{i_k} \in \mathfrak{h}_{\mathbb{R}}.$$

From definition (5.2) of the sets $\overline{\Delta}_i$ we obtain that for $\alpha \in \overline{\Delta}_{i_k}$

$$(5.5) \quad \bar{h}(\alpha) = \sum_{j \leq k} h_{i_j}(\alpha) = h_{i_k}(\alpha) + \sum_{j < k} h_{i_j}(\alpha)$$

Now condition (5.3), the previous identity and the inequality $|x + y| \geq |x| - |y|$ imply that for $\alpha \in \overline{\Delta}_{i_k}$ we have

$$|\bar{h}(\alpha)| \geq |h_{i_k}(\alpha)| - \left| \sum_{j < k} h_{i_j}(\alpha) \right| > 0.$$

Since Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$, $\Delta = \bigcup_{k=0}^M \overline{\Delta}_{i_k}$, the last inequality ensures that \bar{h} belongs to a Weyl chamber of the root system Δ , and one can define the subset of positive roots Δ_+ and the set of simple positive roots Γ with respect to that chamber. From condition (5.3) and formula (5.5) we also obtain that a root $\alpha \in \overline{\Delta}_{i_k}$ is positive if and only if

$$(5.6) \quad h_{i_k}(\alpha) > 0.$$

We denote by $(\overline{\Delta}_{i_k})_+$ the set of positive roots contained in $\overline{\Delta}_{i_k}$, $(\overline{\Delta}_{i_k})_+ = \Delta_+ \cap \overline{\Delta}_{i_k}$.

We shall also need a parabolic subalgebra \mathfrak{p} of \mathfrak{g} associated to the semisimple element $\bar{h}_0 = \sum_{k=1}^M h_{i_k} \in \mathfrak{h}_{\mathbb{R}}$ associated to $s \in W$. This subalgebra is defined with the help of the linear eigenspace decomposition of \mathfrak{g} with respect to the adjoint action of \bar{h}_0 on \mathfrak{g} , $\mathfrak{g} = \bigoplus_m (\mathfrak{g})_m$, $(\mathfrak{g})_m = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = mx\}$, $m \in \mathbb{R}$. By definition $\mathfrak{p} = \bigoplus_{m \geq 0} (\mathfrak{g})_m$ is a parabolic subalgebra in \mathfrak{g} , $\mathfrak{n} = \bigoplus_{m > 0} (\mathfrak{g})_m$ and $\mathfrak{l} = \{x \in \mathfrak{g} \mid [\bar{h}_0, x] = 0\}$ are the nilradical and the Levi factor of \mathfrak{p} , respectively. Note that we have natural inclusions of Lie algebras $\mathfrak{p} \supset \mathfrak{b}_+ \supset \mathfrak{n}$, where \mathfrak{b}_+ is the Borel subalgebra of \mathfrak{g} corresponding to the system Γ of simple roots, and Δ_{i_0} is the root system of the reductive Lie algebra \mathfrak{l} .

For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $l(w)$ of the element w with respect to the system Γ of simple roots in Δ_+ .

Now recall that in the classification theory of conjugacy classes in the Weyl group W of the complex simple Lie algebra \mathfrak{g} the so-called primitive (or semi-Coxeter in another terminology) elements play a primary role. The primitive elements $w \in W$ are characterized by the property $\det(1 - w) = \det a$, where a is the Cartan matrix of \mathfrak{g} . According to the results of [6] the element s of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ is primitive in the Weyl group W' of a regular semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the form

$$\mathfrak{g}' = \mathfrak{h}' + \sum_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

where Δ' is a root subsystem of the root system Δ of \mathfrak{g} , \mathfrak{g}_α is the root subspace of \mathfrak{g} corresponding to root α , and \mathfrak{h}' is a Lie subalgebra of \mathfrak{h} (it coincides with \mathfrak{h}' introduced in Section 4).

Moreover, by Theorem C in [6] s can be represented as a product of two involutions,

$$(5.7) \quad s = s^1 s^2,$$

where $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \dots, \gamma_{l'}$ form a linear basis of \mathfrak{h}' , in particular l' is the rank of \mathfrak{g}' .

Proposition 5.1. ([32], **Peoposition 5.1**) *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots defined with the help of element (5.4), $\Delta_+ = \{\alpha \in \Delta | \bar{h}(\alpha) > 0\}$.*

Then there is a normal ordering of the root system Δ_+ of the following form

$$(5.8) \quad \begin{aligned} & \beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\ & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \\ & \beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\ & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2, \\ & \beta_1^0, \dots, \beta_{D_0}^0, \end{aligned}$$

where

$$\begin{aligned} & \{\beta_1^1, \dots, \beta_t^1, \beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\ & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1\} = \Delta_{s^1}, \\ & \{\beta_{t+1}^1, \dots, \beta_{t+\frac{p-n}{2}}^1, \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \\ & \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \gamma_3, \dots, \gamma_n\} = \{\alpha \in \Delta_+ | s^1(\alpha) = -\alpha\}, \\ & \{\beta_1^2, \dots, \beta_q^2, \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\ & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2, \beta_{2q+2m_{l(s^2)}-(l'-n)+1}^2, \dots, \beta_{l(s^2)}^2\} = \Delta_{s^2}, \\ & \{\gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \\ & \gamma_{l'}, \beta_{q+m_{l(s^2)}+1}^2, \dots, \beta_{2q+2m_{l(s^2)}-(l'-n)}^2\} = \{\alpha \in \Delta_+ | s^2(\alpha) = -\alpha\}, \\ & \{\beta_1^0, \dots, \beta_{D_0}^0\} = \Delta_0 = \{\alpha \in \Delta_+ | s(\alpha) = \alpha\}, \end{aligned}$$

and s^1, s^2 are the involutions entering decomposition (5.7), $s^1 = s_{\gamma_1} \dots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \dots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \dots, \gamma_n$ and $\gamma_{n+1}, \dots, \gamma_{l'}$ are positive and mutually orthogonal.

The length of the ordered segment $\Delta_{\mathbf{m}+} \subset \Delta$ in normal ordering (5.8),

$$(5.9) \quad \begin{aligned} \Delta_{\mathbf{m}+} = & \gamma_1, \beta_{t+\frac{p-n}{2}+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_1}^1, \gamma_2, \beta_{t+\frac{p-n}{2}+n_1+2}^1, \dots, \beta_{t+\frac{p-n}{2}+n_2}^1, \\ & \gamma_3, \dots, \gamma_n, \beta_{t+p+1}^1, \dots, \beta_{l(s^1)}^1, \dots, \beta_1^2, \dots, \beta_q^2, \\ & \gamma_{n+1}, \beta_{q+2}^2, \dots, \beta_{q+m_1}^2, \gamma_{n+2}, \beta_{q+m_1+2}^2, \dots, \beta_{q+m_2}^2, \gamma_{n+3}, \dots, \gamma_{l'}, \end{aligned}$$

is equal to

$$(5.10) \quad D - \left(\frac{l(s) - l'}{2} + D_0 \right),$$

where D is the number of roots in Δ_+ , $l(s)$ is the length of s and D_0 is the number of positive roots fixed by the action of s .

Moreover, for any two roots $\alpha, \beta \in \Delta_{\mathbf{m}+}$ such that $\alpha < \beta$ the sum $\alpha + \beta$ cannot be represented as a linear combination $\sum_{k=1}^q c_k \gamma_{i_k}$, where $c_k \in \mathbb{N}$ and $\alpha < \gamma_{i_1} < \dots < \gamma_{i_q} < \beta$.

We call the system of positive roots Δ_+ ordered as in (5.8) the normally ordered system of positive roots associated to the (conjugacy class) of the Weyl group element $s \in W$. We shall also need the circular ordering in the root system Δ corresponding to normal ordering (5.8) of the positive root system Δ_+ .

Let $\beta_1, \beta_2, \dots, \beta_D$ be a normal ordering of a positive root system Δ_+ . Then following [18] one can introduce the corresponding circular normal ordering of the root system Δ where the roots in Δ are located on a circle in the following way

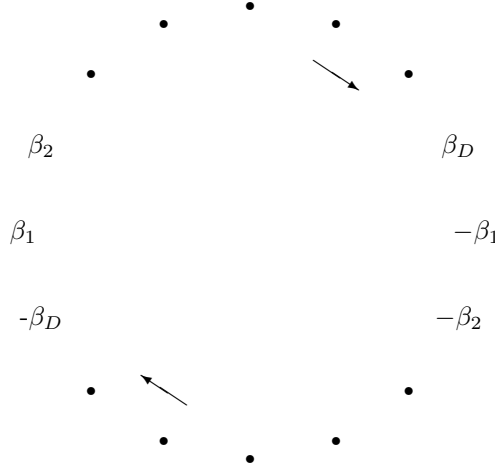


Fig.2

Let $\alpha, \beta \in \Delta$. One says that the segment $[\alpha, \beta]$ of the circle is minimal if it does not contain the opposite roots $-\alpha$ and $-\beta$ and the root β follows after α on the circle above, the circle being oriented clockwise. In that case one also says that $\alpha < \beta$ in the sense of the circular normal ordering,

$$(5.11) \quad \alpha < \beta \Leftrightarrow \text{the segment } [\alpha, \beta] \text{ of the circle is minimal.}$$

Later we shall need the following property of minimal segments which is a direct consequence of Proposition 3.3 in [17].

Lemma 5.2. *Let $[\alpha, \beta]$ be a minimal segment in a circular normal ordering of a root system Δ . Then if $\alpha + \beta$ is a root we have*

$$\alpha < \alpha + \beta < \beta.$$

Now following [32] we define the subalgebras of $U_h(\mathfrak{g})$ which resemble nilpotent subalgebras in \mathfrak{g} and possess nontrivial characters.

Theorem 5.3. ([32], **Theorem 6.1**) *Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$, Δ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Fix a decomposition (5.7) of s and let Δ_+ be the system of positive roots associated to s . Let $U_h^s(\mathfrak{g})$ be the realization of the quantum group $U_h(\mathfrak{g})$ associated to s . Let $e_\beta \in U_h^s(\mathfrak{n}_+)$, $\beta \in \Delta_+$ be the root vectors associated to the corresponding normal ordering (5.8) of Δ_+ .*

Then elements $e_\beta \in U_h^s(\mathfrak{n}_+)$, $\beta \in \Delta_{\mathfrak{m}_+}$, where $\Delta_{\mathfrak{m}_+} \subset \Delta$ is ordered segment (5.9), generate a subalgebra $U_h^s(\mathfrak{m}_+) \subset U_h^s(\mathfrak{g})$ such that $U_h^s(\mathfrak{m}_+)/hU_h^s(\mathfrak{m}_+) \simeq U(\mathfrak{m}_+)$, where \mathfrak{m}_+ is the Lie subalgebra of \mathfrak{g} generated by the root vectors X_α , $\alpha \in \Delta_{\mathfrak{m}_+}$. The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \dots e_{\beta_D}^{r_D}$, $r_i \in \mathbb{N}$, $i = 1, \dots, D$ and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, form a topological basis of $U_h^s(\mathfrak{m}_+)$.

Moreover the map $\chi_h^s : U_h^s(\mathfrak{m}_+) \rightarrow \mathbb{C}[[h]]$ defined on generators by

$$(5.12) \quad \chi_h^s(e_\beta) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ c_i & \beta = \gamma_i, c_i \in \mathbb{C}[[h]] \end{cases}$$

is a character of $U_h^s(\mathfrak{m}_+)$.

Lemma 5.4. ([32], **Lemma 6.2**) *Let $P_{\mathfrak{h}'^*}$ be the orthogonal projection operator onto \mathfrak{h}'^* in \mathfrak{h}^* , with respect to the Killing form. Then the matrix elements of the operator $\frac{1+s}{1-s}P_{\mathfrak{h}'^*}$ in the basis $\gamma_1, \dots, \gamma_{l'}$ are of the form:*

$$(5.13) \quad \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} \gamma_i, \gamma_j \right) = \varepsilon_{ij}(\gamma_i, \gamma_j),$$

where

$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

The matrix A_{ij} is called the Carter matrix of s . We shall also use the Lie subalgebra \mathfrak{m}_- of \mathfrak{g} generated by the root vectors $X_{-\alpha}$, $\alpha \in \Delta_{\mathfrak{m}_+}$.

6. SOME SPECIALIZATIONS OF THE ALGEBRA $U_h^s(\mathfrak{g})$

In this section we introduce some forms of the quantum group $U_h^s(\mathfrak{g})$ which are similar to the rational form, the restricted integral form and to its specialization for the standard quantum group $U_h(\mathfrak{g})$. The motivations of the definitions given below will be clear in Section 9. The results in this section are slight modifications of similar statements for $U_h(\mathfrak{g})$, and we refer to [7], Ch. 9 for the proofs.

We start with the observation that the numbers

$$(6.1) \quad p_{ij} = \left(\frac{1+s}{1-s} P_{\mathfrak{h}'^*} Y_i, Y_j \right) + (Y_i, Y_j)$$

are rational, $p_{ij} \in \mathbb{Q}$.

Indeed, let γ_i^* , $i = 1, \dots, l'$ be the basis of \mathfrak{h}'^* dual to γ_i , $i = 1, \dots, l'$ with respect to the restriction of the bilinear form (\cdot, \cdot) to \mathfrak{h}'^* . Since the numbers (γ_i, γ_j) are integer each element γ_i^* has the form

$\gamma_i^* = \sum_{j=1}^{l'} m_{ij} \gamma_j$, where $m_{ij} \in \mathbb{Q}$. Now we have

$$\begin{aligned} & \left(\frac{1+s}{1-s} P_{\mathfrak{h}'} Y_i, Y_j \right) + (Y_i, Y_j) = \\ & = \sum_{k,l,p,q=1}^{l'} \gamma_k(Y_i) \gamma_l(Y_j) \left(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_p, \gamma_q \right) m_{kp} m_{lq} + (Y_i, Y_j). \end{aligned}$$

All the terms in the r.h.s. of the last identity are rational since $\gamma_i(Y_j) \in \mathbb{Z}$ for any $i = 1, \dots, l'$ and $j = 1, \dots, l$ because Y_i are the fundamental weights, the numbers $\left(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \gamma_p, \gamma_q \right)$ are integer by Lemma 5.4, the coefficients m_{ij} are rational as we observed above, and the scalar products (Y_i, Y_j) of the fundamental weights are rational. Therefore the numbers p_{ij} are rational.

Denote by d the smallest integer number divisible by all the denominators of the rational numbers $p_{ij}/2$, $i, j = 1, \dots, l$.

Let $U_q^s(\mathfrak{g})$ be the $\mathbb{C}(q^{\frac{1}{2d}})$ -subalgebra of $U_h^s(\mathfrak{g})$ generated by the elements $e_i, f_i, t_i^{\pm 1} = \exp(\pm \frac{h}{2d} H_i)$, $i = 1, \dots, l$.

The defining relations for the algebra $U_q^s(\mathfrak{g})$ are

$$\begin{aligned} & t_i t_j = t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i e_j t_i^{-1} = q^{\frac{a_{ij}}{2d}} e_j, \quad t_i f_j t_i^{-1} = q^{-\frac{a_{ij}}{2d}} f_j, \\ & e_i f_j - q^{c_{ij}} f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad c_{ij} = \left(\frac{1+s}{1-s} P_{\mathfrak{h}'*} \alpha_i, \alpha_j \right) \\ & K_i = t_i^{2dd_i}, \\ (6.2) \quad & \sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (e_i)^{1-a_{ij}-r} e_j (e_i)^r = 0, \quad i \neq j, \\ & \sum_{r=0}^{1-a_{ij}} (-1)^r q^{rc_{ij}} \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (f_i)^{1-a_{ij}-r} f_j (f_i)^r = 0, \quad i \neq j. \end{aligned}$$

Note that by the choice of d we have $q^{c_{ij}} \in \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$.

The second form of $U_h^s(\mathfrak{g})$ is a subalgebra $U_{\mathcal{A}}^s(\mathfrak{g})$ in $U_q^s(\mathfrak{g})$ over the ring $\mathcal{A} = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}, \frac{1}{[2]_{q_i}}, \dots, \frac{1}{[r]_{q_i}}]$, where $i = 1, \dots, l$, r is the maximal number k_i that appears in the right-hand sides of formulas (3.12) for various α and β . $U_{\mathcal{A}}^s(\mathfrak{g})$ is the subalgebra in $U_q^s(\mathfrak{g})$ generated over \mathcal{A} by the elements $t_i^{\pm 1}, \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, e_i, f_i, i = 1, \dots, l$.

The most important for us is the specialization $U_{\varepsilon}^s(\mathfrak{g})$ of $U_{\mathcal{A}}^s(\mathfrak{g})$, $U_{\varepsilon}^s(\mathfrak{g}) = U_{\mathcal{A}}^s(\mathfrak{g}) / (q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}}) U_{\mathcal{A}}^s(\mathfrak{g})$, $\varepsilon \in \mathbb{C}^*$, $[r]_{\varepsilon_i}! \neq 0$, $i = 1, \dots, l$. Note that $[r]_1! \neq 0$, and hence one can define the specialization $U_1^s(\mathfrak{g})$.

$U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_{\varepsilon}^s(\mathfrak{g})$ are Hopf algebras with the comultiplication induced from $U_h^s(\mathfrak{g})$.

If in addition $\varepsilon^{2d_i} \neq 1$, $i = 1, \dots, l$, then $U_{\varepsilon}^s(\mathfrak{g})$ is generated over \mathbb{C} by $t_i^{\pm 1}, e_i, f_i, i = 1, \dots, l$ subject to relations (6.2) where $q = \varepsilon$.

The algebra $U_{\mathcal{A}}^s(\mathfrak{g})$ has a similar description.

The elements t_i are central in the algebra $U_1^s(\mathfrak{g})$, and the quotient of $U_1^s(\mathfrak{g})$ by the two-sided ideal generated by $t_i - 1$ is isomorphic to $U(\mathfrak{g})$.

If V is a $U_q^s(\mathfrak{g})$ -module then its weight spaces are all the nonzero $\mathbb{C}(q^{\frac{1}{2d}})$ -linear subspaces of the form

$$V_{\mathbf{c}} = \{v \in V, t_i v = c_i v, c_i \in \mathbb{C}(q^{\frac{1}{2d}})^*, i = 1, \dots, l\}.$$

The l -tuple $\mathbf{c} = (c_1, \dots, c_l) \in (\mathbb{C}(q^{\frac{1}{2d}})^*)^l$ is called a weight.

If $\mathbf{c}' = (c'_1, \dots, c'_l)$ is another weight one says that $\mathbf{c}' \leq \mathbf{c}$ if $c'_i c_i = q^{\frac{1}{2d}\beta(H_i)}$ for some $\beta \in Q^+ = \bigoplus_{i=1}^l \mathbb{N}\alpha_i$ and all $i = 1, \dots, l$.

A highest weight $U_q^s(\mathfrak{g})$ -module is a $U_q^s(\mathfrak{g})$ -module V which contains a weight vector $v \in V_{\mathbf{c}}$ annihilated by the action of all elements e_i and such that $V = U_q^s(\mathfrak{g})v$. In that case we also have a weight space decomposition

$$V = \bigoplus_{\mathbf{c}' \leq \mathbf{c}} V_{\mathbf{c}'},$$

and $\dim_{\mathbb{C}(q^{\frac{1}{2d}})} V_{\mathbf{c}} = 1$. In particular, \mathbf{c} is uniquely defined by V . It is called the highest weight of V , and v is called the highest weight vector.

Verma and finite-dimensional irreducible $U_q^s(\mathfrak{g})$ -modules are defined in the usual way. For instance, the Verma module $M_q(\lambda)$ of highest weight λ corresponding to a character $\lambda : U_q^s(\mathfrak{h}) \rightarrow \mathbb{C}(q^{\frac{1}{2d}})^*$ is the quotient of $U_q^s(\mathfrak{g})$ by the right ideal generated by e_i and $t_i - \lambda(t_i)$, where $i = 1, \dots, l$. We frequently and formally write λ in the exponential form, $\lambda(t_i) = q^{\frac{1}{2d}\lambda(H_i)}$, $i = 1, \dots, l$. In particular, if $\mu \in P = \{\eta \in \mathfrak{h}^*, \eta(H_i) \in \mathbb{Z} \text{ for all } i\}$ we shall denote by $\lambda - \mu$ the character of $U_q^s(\mathfrak{h})$ such that $(\lambda - \mu)(t_i) = q^{\frac{1}{2d}(\lambda - \mu)(H_i)} = \lambda(t_i)q^{\frac{1}{2d}(-\mu)(H_i)}$, $i = 1, \dots, l$.

The image of 1 in $M_q(\lambda)$ is the highest weight vector v_λ in $M_q(\lambda)$. For $\lambda \in P_+ = \{\mu \in \mathfrak{h}^*, \mu(H_i) \in \mathbb{N} \text{ for all } i\}$ the unique irreducible quotient $V_q(\lambda)$ of $M_q(\lambda)$ is a finite-dimensional irreducible representation of $U_q^s(\mathfrak{g})$.

If V is a highest weight $U_q(\mathfrak{g})$ -module with highest weight vector v then $V_{\mathcal{A}} = U_{\mathcal{A}}^s(\mathfrak{g})v$ is a $U_{\mathcal{A}}^s$ -submodule of V which has weight decomposition induced by that of V .

Moreover, $V_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q^{\frac{1}{2d}}) \simeq V$, $V_{\mathcal{A}}$ is the direct sum of its intersections with the weight spaces of V , each such intersection is a free \mathcal{A} -module of finite rank, and $\overline{V} = V_{\mathcal{A}}/(q^{\frac{1}{2d}} - 1)V_{\mathcal{A}}$ is naturally a $U(\mathfrak{g})$ -module. In particular for $\lambda \in P_+$, $M(\lambda) = \overline{M}_q(\lambda)$ and $V(\lambda) = \overline{V}_q(\lambda)$ are the Verma and the finite-dimensional irreducible $U(\mathfrak{g})$ -modules with highest weight λ .

For Verma and finite-dimensional representations of highest weight μ every nonzero weight subspace has weight of the form $(q^{\frac{1}{2d}\lambda(H_1)}, \dots, q^{\frac{1}{2d}\lambda(H_l)})$, where $\lambda = \mu - \nu$, $\nu \in P_+ = \{\eta \in \mathfrak{h}^*, \eta(H_i) \in \mathbb{N} \text{ for all } i\}$. One simply calls such a subspace a subspace of weight λ .

Similarly one can define highest weight, Verma and finite-dimensional $U_\varepsilon^s(\mathfrak{g})$ -modules in case when ε is transcendental; one should just replace q with ε in the definitions above for the algebra $U_q^s(\mathfrak{g})$.

For the solution $n_{ij} = \frac{1}{2d_j}c_{ij}$ to equations (4.2) the root vectors e_β, f_β belong to all the above introduced subalgebras of $U_h(\mathfrak{g})$, and one can define Poincaré-Birkhoff-Witt bases for them in a similar way. From now on we shall assume that the solution to equations (4.2) is fixed as above, $n_{ij} = \frac{1}{2d_j}c_{ij}$.

If we define for $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$

$$t^{\mathbf{s}} = t_1^{s_1} \dots t_l^{s_l}$$

and denote by $U_q^s(\mathfrak{n}_+)$, $U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$ the subalgebras of $U_q^s(\mathfrak{g})$ generated by the e_i , f_i and by the t_i , respectively, then the elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \dots e_{\beta_D}^{r_D}$, $f^{\mathbf{t}} = f_{\beta_D}^{t_D} \dots f_{\beta_1}^{t_1}$ and $t^{\mathbf{s}}$, for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$, $\mathbf{s} \in \mathbb{Z}^l$, form bases of $U_q^s(\mathfrak{n}_+)$, $U_q^s(\mathfrak{n}_-)$ and $U_q^s(\mathfrak{h})$, respectively, and the products $e^{\mathbf{r}} t^{\mathbf{s}} f^{\mathbf{t}}$ form a basis of $U_q^s(\mathfrak{g})$. In particular, multiplication defines an isomorphism:

$$U_q^s(\mathfrak{n}_-) \otimes U_q^s(\mathfrak{h}) \otimes U_q^s(\mathfrak{n}_+) \rightarrow U_q^s(\mathfrak{g}).$$

We shall also use quantum analogues of Borel subalgebras $U_q^s(\mathfrak{b}_+)$, $U_q^s(\mathfrak{b}_-)$, $U_q^s(\mathfrak{b}_\pm)$ is the subalgebra in $U_q^s(\mathfrak{g})$ generated by $U_q^s(\mathfrak{n}_\pm)$ and by $U_q^s(\mathfrak{h})$, $U_q^s(\mathfrak{b}_\pm) = U_q^s(\mathfrak{n}_+)U_q^s(\mathfrak{h})$.

By specializing the above constructed basis for $q = \varepsilon$ we obtain a similar basis and similar subalgebras for $U_\varepsilon^s(\mathfrak{g})$.

Let $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$ be the subalgebras of $U_{\mathcal{A}}^s(\mathfrak{g})$ generated by the e_i and by the $f_i, i = 1, \dots, l$, respectively. Using the root vectors e_β and f_β we can construct a basis of $U_{\mathcal{A}}^s(\mathfrak{g})$. Namely, the elements $e^{\mathbf{r}}, f^{\mathbf{t}}$ for $\mathbf{r}, \mathbf{t} \in \mathbb{N}^N$ form bases of $U_{\mathcal{A}}^s(\mathfrak{n}_+), U_{\mathcal{A}}^s(\mathfrak{n}_-)$, respectively.

The elements

$$\left[\begin{matrix} K_i; c \\ r \end{matrix} \right]_{q_i} = \prod_{s=1}^r \frac{K_i q_i^{c+1-s} - K_i^{-1} q_i^{s-1-c}}{q_i^s - q_i^{-s}}, \quad i = 1, \dots, l, \quad c \in \mathbb{Z}, \quad r \in \mathbb{N}$$

belong to $U_{\mathcal{A}}^s(\mathfrak{g})$. Denote by $U_{\mathcal{A}}^s(\mathfrak{h})$ the subalgebra of $U_{\mathcal{A}}^s(\mathfrak{g})$ generated by those elements and by $t_i^{\pm 1}, i = 1, \dots, l$.

Then multiplication defines an isomorphism of \mathcal{A} modules:

$$U_{\mathcal{A}}^s(\mathfrak{n}_-) \otimes U_{\mathcal{A}}^s(\mathfrak{h}) \otimes U_{\mathcal{A}}^s(\mathfrak{n}_+) \rightarrow U_{\mathcal{A}}^s(\mathfrak{g}).$$

We shall also use the subalgebras $U_{\mathcal{A}}^s(\mathfrak{b}_{\pm}) \subset U_{\mathcal{A}}^s(\mathfrak{g})$ generated by $U_{\mathcal{A}}^s(\mathfrak{n}_{\pm})$ and by $U_{\mathcal{A}}^s(\mathfrak{h})$.

A basis for $U_{\mathcal{A}}^s(\mathfrak{h})$ is a little bit more difficult to describe. We do not need its explicit description (see [7], Proposition 9.3.3 for details).

None of the subalgebras of $U_{\mathfrak{h}}^s(\mathfrak{g})$ introduced above is quasitriangular. However, one can define an action of R-matrix (4.10) in the finite-dimensional representations of $U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_\varepsilon^s(\mathfrak{g})$. Indeed, observe that one can write R-matrix (4.10) in the factorized form

$$(6.3) \quad \mathcal{R}^s = \mathcal{E} \tilde{\mathcal{R}},$$

where

$$\mathcal{E} = \exp \left[h \left(\sum_{i=1}^l (Y_i \otimes H_i) + \sum_{i=1}^l \frac{1+s}{1-s} P_{\mathfrak{h}'} H_i \otimes Y_i \right) \right]$$

and

$$\tilde{\mathcal{R}} = \sum_{u_1, \dots, u_D=0}^{\infty} \prod_{r=1}^D q_{\beta_r}^{\frac{1}{2} u_r (u_r + 1)} (1 - q_{\beta_r}^{-2})^{u_r} e_{\beta_r}^{u_r} \otimes e^{u_r h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee} (f_{\beta_r})^{(u_r)},$$

where the order of the factors in the product is such that the β_r -term appears to the right of the β_s -term if $r > s$.

Using the fact that the numbers p_{ij} defined by (6.1) are of the form $p_{ij} = \frac{2v_{ij}}{d}$, $v_{ij} \in \mathbb{Z}$ one can check that actually $e^{u_r h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee} \in U_{\mathcal{A}}^s(\mathfrak{g})$. Therefore $e_{\beta_r}^{u_r} \otimes e^{u_r h \frac{1+s}{1-s} P_{\mathfrak{h}'} \beta^\vee} (f_{\beta_r})^{(u_r)} \in U_{\mathcal{A}}^s(\mathfrak{g}) \otimes U_{\mathcal{A}}^s(\mathfrak{g})$.

For every two finite-dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$ -modules V and W only finitely many terms in the expression for $\tilde{\mathcal{R}}$ act nontrivially on $V \otimes W$ since the action of root vectors on V and W is nilpotent. Therefore the action of the element $\tilde{\mathcal{R}}$ in the space $V \otimes W$ is well defined.

Moreover, if V_μ and W_λ are two weight subspaces of V and W of weights $\mu, \lambda \in P$ then one can define an action of \mathcal{E} in $V_\mu \otimes W_\lambda$ as multiplication by the scalar $q^{(\lambda, \mu) + (\frac{1+s}{1-s} P_{\mathfrak{h}'} \lambda, \mu)}$. Since the numbers p_{ij} defined by (6.1) are of the form $p_{ij} = \frac{2v_{ij}}{d}$ this scalar is an element of $\mathcal{A} = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}]$.

If we define an action of the element \mathcal{R}^s in $V \otimes W$ as the composition of the above defined action of the operators \mathcal{E} and $\tilde{\mathcal{R}}$ in $V \otimes W$ and denote the obtained operator by $R^{V,W}$ then one can check that

$$R^{V,W} (\pi_V \otimes \pi_W) \Delta_s(x) R^{V,W-1} = (\pi_W \otimes \pi_V) \Delta_s^{opp}(x),$$

where π_V, π_W are the representations V and W and Δ_s is the comultiplication on $U_{\mathcal{A}}^s(\mathfrak{g})$. Moreover, $R^{V,W}$ satisfies the quantum Yang-Baxter equation.

By specializing q to a particular value $q = \varepsilon$ one can obtain an operator with similar properties acting in the tensor product of any two finite-dimensional $U_\varepsilon^s(\mathfrak{g})$ -modules. Obviously, the above construction can be applied in case of the algebra $U_q^s(\mathfrak{g})$ as well.

Finally we discuss an obvious analogue of the subalgebra $U_h^s(\mathfrak{m}_+) \subset U_h^s(\mathfrak{g})$ for $U_{\mathcal{A}}^s(\mathfrak{g})$.

Let $U_{\mathcal{A}}^s(\mathfrak{m}_+) \subset U_{\mathcal{A}}^s(\mathfrak{g})$ be the subalgebra generated by elements $e_{\beta} \in U_{\mathcal{A}}^s(\mathfrak{n}_+)$, $\beta \in \Delta_{\mathfrak{m}_+}$, where $\Delta_{\mathfrak{m}_+} \subset \Delta$ is the ordered segment (5.9). The defining relations in the subalgebra $U_{\mathcal{A}}^s(\mathfrak{m}_+)$ are given by formula (4.6),

$$(6.4) \quad e_{\alpha}e_{\beta} - q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)}e_{\beta}e_{\alpha} = \sum_{\alpha < \gamma_1 < \dots < \gamma_n < \beta} C'(k_1, \dots, k_n) e_{\gamma_1}^{k_1} e_{\gamma_2}^{k_2} \dots e_{\gamma_n}^{k_n},$$

where $C'(k_1, \dots, k_n) \in \mathcal{A}$.

The elements $e^{\mathbf{r}} = e_{\beta_1}^{r_1} \dots e_{\beta_D}^{r_D}$, $r_i \in \mathbb{N}$, $i = 1, \dots, D$, and r_i can be strictly positive only if $\beta_i \in \Delta_{\mathfrak{m}_+}$, form a basis of $U_{\mathcal{A}}^s(\mathfrak{m}_+)$.

Obviously $U_{\mathcal{A}}^s(\mathfrak{m}_+)/(q^{\frac{1}{2d}} - 1)U_{\mathcal{A}}^s(\mathfrak{m}_+) \simeq U(\mathfrak{m}_+)$, where \mathfrak{m}_+ is the Lie subalgebra of \mathfrak{g} generated by the root vectors X_{α} , $\alpha \in \Delta_{\mathfrak{m}_+}$.

Moreover, the map $\chi_{\mathcal{A}}^s : U_{\mathcal{A}}^s(\mathfrak{m}_+) \rightarrow \mathcal{A}$ defined on generators by

$$\chi_{\mathcal{A}}^s(e_{\beta}) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_n\} \\ c_i & \beta = \gamma_i, c_i \in \mathcal{A} \end{cases}$$

is a character of $U_{\mathcal{A}}^s(\mathfrak{m}_+)$.

By specializing q to a particular value $q = \varepsilon$ one can obtain a subalgebra $U_{\varepsilon}^s(\mathfrak{m}_+) \subset U_{\varepsilon}^s(\mathfrak{g})$ with similar properties.

The algebras $U_q^s(\mathfrak{g})$, $U_{\mathcal{A}}^s(\mathfrak{g})$ and $U_{\varepsilon}^s(\mathfrak{g})$ can be equipped with remarkable filtrations such that the associated graded algebras are almost commutative (see [8]). For $\mathbf{r}, \mathbf{t} \in \mathbb{N}^D$ define the height of the element $u_{\mathbf{r},\mathbf{t},t} = e^{\mathbf{r}} t f^{\mathbf{t}}$, $t \in U_q^s(\mathfrak{h})$ as follows $\text{ht}(u_{\mathbf{r},\mathbf{t},t}) = \sum_{i=1}^D (t_i + r_i) \text{ht } \beta_i \in \mathbb{N}$, where $\text{ht } \beta_i$ is the height of the root β_i . Introduce also the degree of $u_{\mathbf{r},\mathbf{t},t}$ by

$$d(u_{\mathbf{r},\mathbf{t},t}) = (r_1, \dots, r_D, t_D, \dots, t_1, \text{ht}(u_{\mathbf{r},\mathbf{t},t})) \in \mathbb{N}^{2D+1}.$$

Equip \mathbb{N}^{2D+1} with the total lexicographic order and denote by $(U_q^s(\mathfrak{g}))_k$ the span of elements $u_{\mathbf{r},\mathbf{t},t}$ with $d(u_{\mathbf{r},\mathbf{t},t}) \leq k$ in $U_q^s(\mathfrak{g})$. Then Proposition 1.7 in [8] implies that $(U_q^s(\mathfrak{g}))_k$ is a filtration of $U_q^s(\mathfrak{g})$ such that the associated graded algebra is the associative algebra over $\mathbb{C}(q)$ with generators e_{α} , f_{α} , $\alpha \in \Delta_+$, $t_i^{\pm 1}$, $i = 1, \dots, l$ subject to the relations

$$(6.5) \quad \begin{aligned} t_i t_j &= t_j t_i, \quad t_i t_i^{-1} = t_i^{-1} t_i = 1, \quad t_i e_{\alpha} t_i^{-1} = q^{\frac{H_i(\alpha)}{2d}} e_{\alpha}, \quad t_i f_{\alpha} t_i^{-1} = q^{-\frac{H_i(\alpha)}{2d}} f_{\alpha}, \\ e_{\alpha} f_{\beta} &= q^{(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)} f_{\beta} e_{\alpha}, \\ e_{\alpha} e_{\beta} &= q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)} e_{\beta} e_{\alpha}, \quad \alpha < \beta, \\ f_{\alpha} f_{\beta} &= q^{(\alpha,\beta)+(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha,\beta)} f_{\beta} f_{\alpha}, \quad \alpha < \beta. \end{aligned}$$

Such algebras are called semi-commutative. A similar result holds for the algebras $U_{\varepsilon}^s(\mathfrak{g})$ and $U_{\mathcal{A}}^s(\mathfrak{g})$.

7. POISSON-LIE GROUPS

In this section we recall some notions concerned with Poisson-Lie groups (see [7], [10], [22], [26]). These facts will be used in Section 9 to define q-W algebras.

Let G be a finite-dimensional Lie group equipped with a Poisson bracket, \mathfrak{g} its Lie algebra. G is called a Poisson-Lie group if the multiplication $G \times G \rightarrow G$ is a Poisson map. A Poisson bracket satisfying this axiom is degenerate and, in particular, is identically zero at the unit element of the group. Linearizing this bracket at the unit element defines the structure of a Lie algebra in the space $T_e^*G \simeq \mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called the tangent bialgebra of G .

Lie brackets in \mathfrak{g} and \mathfrak{g}^* satisfy the following compatibility condition:

Let $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ be the dual of the commutator map $[\cdot, \cdot]_* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. Then δ is a 1-cocycle on \mathfrak{g} (with respect to the adjoint action of \mathfrak{g} on $\mathfrak{g} \wedge \mathfrak{g}$).

Let c_{ij}^k, f_c^{ab} be the structure constants of $\mathfrak{g}, \mathfrak{g}^*$ with respect to the dual bases $\{e_i\}, \{e^i\}$ in $\mathfrak{g}, \mathfrak{g}^*$. The compatibility condition means that

$$c_{ab}^s f_s^{ik} - c_{as}^i f_b^{sk} + c_{as}^k f_b^{si} - c_{bs}^k f_a^{si} + c_{bs}^i f_a^{sk} = 0.$$

This condition is symmetric with respect to exchange of c and f . Thus if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, then $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra.

The following proposition shows that the category of finite-dimensional Lie bialgebras is isomorphic to the category of finite-dimensional connected simply connected Poisson-Lie groups.

Proposition 7.1. ([7], Theorem 1.3.2) *If G is a connected simply connected finite-dimensional Lie group, every bialgebra structure on \mathfrak{g} is the tangent bialgebra of a unique Poisson structure on G which makes G into a Poisson-Lie group.*

Let G be a finite-dimensional Poisson-Lie group, $(\mathfrak{g}, \mathfrak{g}^*)$ the tangent bialgebra of G . The connected simply connected finite-dimensional Poisson-Lie group corresponding to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called the dual Poisson-Lie group and denoted by G^* .

$(\mathfrak{g}, \mathfrak{g}^*)$ is called a factorizable Lie bialgebra if the following conditions are satisfied (see [10], [22]):

- (1) \mathfrak{g} is equipped with a non-degenerate invariant scalar product (\cdot, \cdot) .

We shall always identify \mathfrak{g}^* and \mathfrak{g} by means of this scalar product.

- (2) The dual Lie bracket on $\mathfrak{g}^* \simeq \mathfrak{g}$ is given by

$$(7.1) \quad [X, Y]_* = \frac{1}{2} ([rX, Y] + [X, rY]), \quad X, Y \in \mathfrak{g},$$

where $r \in \text{End } \mathfrak{g}$ is a skew symmetric linear operator (classical r -matrix).

- (3) r satisfies the modified classical Yang-Baxter identity:

$$(7.2) \quad [rX, rY] - r([rX, Y] + [X, rY]) = -[X, Y], \quad X, Y \in \mathfrak{g}.$$

Define operators $r_{\pm} \in \text{End } \mathfrak{g}$ by

$$r_{\pm} = \frac{1}{2} (r \pm id).$$

We shall need some properties of the operators r_{\pm} . Denote by \mathfrak{b}_{\pm} and \mathfrak{n}_{\mp} the image and the kernel of the operator r_{\pm} :

$$(7.3) \quad \mathfrak{b}_{\pm} = \text{Im } r_{\pm}, \quad \mathfrak{n}_{\mp} = \text{Ker } r_{\pm}.$$

Proposition 7.2. ([4], [24]) *Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a factorizable Lie bialgebra. Then*

- (i) $\mathfrak{b}_{\pm} \subset \mathfrak{g}$ is a Lie subalgebra, the subspace \mathfrak{n}_{\pm} is a Lie ideal in \mathfrak{b}_{\pm} , $\mathfrak{b}_{\pm}^{\perp} = \mathfrak{n}_{\pm}$.

- (ii) \mathfrak{n}_{\pm} is an ideal in \mathfrak{g}^* .

- (iii) \mathfrak{b}_{\pm} is a Lie subalgebra in \mathfrak{g}^* . Moreover $\mathfrak{b}_{\pm} = \mathfrak{g}^* / \mathfrak{n}_{\pm}$.

- (iv) $(\mathfrak{b}_{\pm}, \mathfrak{b}_{\pm}^*)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{b}_{\pm}, \mathfrak{b}_{\pm}^*) \simeq (\mathfrak{b}_{\pm}, \mathfrak{b}_{\mp}^*)$. The canonical pairing between \mathfrak{b}_{\mp} and \mathfrak{b}_{\pm} is given by

$$(7.4) \quad (X_{\mp}, Y_{\pm})_{\pm} = (X_{\mp}, r_{\pm}^{-1} Y_{\pm}), \quad X_{\mp} \in \mathfrak{b}_{\mp}; \quad Y_{\pm} \in \mathfrak{b}_{\pm}.$$

The classical Yang-Baxter equation implies that r_{\pm} , regarded as a mapping from \mathfrak{g}^* into \mathfrak{g} , is a Lie algebra homomorphism. Moreover, $r_+^* = -r_-$, and $r_+ - r_- = id$.

Put $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ (direct sum of two copies). The mapping

$$(7.5) \quad \mathfrak{g}^* \rightarrow \mathfrak{d} : X \mapsto (X_+, X_-), \quad X_{\pm} = r_{\pm} X$$

is a Lie algebra embedding. Thus we may identify \mathfrak{g}^* with a Lie subalgebra in \mathfrak{d} .

Naturally, embedding (7.5) extends to a homomorphism

$$G^* \rightarrow G \times G, L \mapsto (L_+, L_-).$$

We shall identify G^* with the corresponding subgroup in $G \times G$.

8. POISSON REDUCTION

In this section we recall basic facts on Poisson reduction (see [35], [25]).

Let M , B , B' be Poisson manifolds. Two Poisson surjections

$$\begin{array}{ccc} & M & \\ \pi' \swarrow & & \searrow \pi \\ B' & & B \end{array}$$

form a dual pair if the pullback $\pi'^*C^\infty(B')$ is the centralizer of $\pi^*C^\infty(B)$ in the Poisson algebra $C^\infty(M)$. In that case the sets $B'_b = \pi'(\pi^{-1}(b))$, $b \in B$ are Poisson submanifolds in B' (see [35]) called reduced Poisson manifolds.

Fix an element $b \in B$. Then the algebra of functions $C^\infty(B'_b)$ may be described as follows. Let I_b be the ideal in $C^\infty(M)$ generated by elements $\pi^*(f)$, $f \in C^\infty(B)$, $f(b) = 0$. Denote $M_b = \pi^{-1}(b)$. Then the algebra $C^\infty(M_b)$ is simply the quotient of $C^\infty(M)$ by I_b . Denote by $P_b : C^\infty(M) \rightarrow C^\infty(M)/I_b = C^\infty(M_b)$ the canonical projection onto the quotient.

Lemma 8.1. *Suppose that the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$. Then one can define an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$ by*

$$(8.1) \quad f \cdot \varphi = P_b(\{\pi^*(f), \tilde{\varphi}\}),$$

where $f \in C^\infty(B)$, $\varphi \in C^\infty(M_b)$ and $\tilde{\varphi} \in C^\infty(M)$ is a representative of φ in $C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Moreover, $C^\infty(B'_b)$ is the subspace of invariants in $C^\infty(M_b)$ with respect to this action.

Proof. Let $\varphi \in C^\infty(M_b)$. Choose a representative $\tilde{\varphi} \in C^\infty(M)$ such that $P_b(\tilde{\varphi}) = \varphi$. Since the map $f \mapsto f(b)$ is a character of the Poisson algebra $C^\infty(B)$, Hamiltonian vector fields of functions $\pi^*(f)$, $f \in C^\infty(B)$ are tangent to the surface M_b . Therefore the r.h.s. of (8.1) only depends on φ but not on the representative $\tilde{\varphi}$, and hence formula (8.1) defines an action of the Poisson algebra $C^\infty(B)$ on the space $C^\infty(M_b)$.

Using the definition of the dual pair we obtain that $\varphi = \pi'^*(\psi)$ for some $\psi \in C^\infty(B'_b)$ if and only if $P_b(\{\pi^*(f), \tilde{\varphi}\}) = 0$ for every $f \in C^\infty(B)$.

Finally we obtain that $C^\infty(B'_b)$ is exactly the subspace of invariants in $C^\infty(M_b)$ with respect to action (8.1). \square

Definition 8.1. *The algebra $C^\infty(B'_b)$ is called a reduced Poisson algebra. We also denote it by $C^\infty(M_b)^{C^\infty(B)}$.*

Remark 8.4. *Note that the description of the algebra $C^\infty(M_b)^{C^\infty(B)}$ obtained in Lemma 8.1 is independent of both the manifold B' and the projection π' . Observe also that the reduced space B'_b may be identified with a cross-section of the action of the Poisson algebra $C^\infty(B)$ on M_b by Hamiltonian vector fields in case when this action is free. In particular, in that case B'_b may be regarded as a submanifold in M_b .*

An important example of dual pairs is provided by Poisson group actions. Recall that a (local) Poisson group action of a Poisson-Lie group A on a Poisson manifold M is a (local) group action $A \times M \rightarrow M$ which is also a Poisson map (as usual, we suppose that $A \times M$ is equipped with the product Poisson structure).

In [25] it is proved that if the space M/A is a smooth manifold, there exists a unique Poisson structure on M/A such that the canonical projection $M \rightarrow M/A$ is a Poisson map.

Let \mathfrak{a} be the Lie algebra of A . Denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between \mathfrak{a}^* and \mathfrak{a} . A map $\mu : M \rightarrow A^*$ is called a moment map for a (local) right Poisson group action $A \times M \rightarrow M$ if (see [20])

$$(8.2) \quad L_{\widehat{X}}\varphi = \langle \mu^*(\theta_{A^*}), X \rangle (\xi_\varphi),$$

where θ_{A^*} is the universal right-invariant Maurer–Cartan form on A^* , $X \in \mathfrak{a}$, \widehat{X} is the corresponding vector field on M and ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(M)$.

By Theorem 4.9 in [20] one can always equip A^* with a Poisson structure in such a way that μ becomes a Poisson mapping. From the definition of the moment map it follows that if M/A is a smooth manifold then the canonical projection $M \rightarrow M/A$ and the moment map $\mu : M \rightarrow A^*$ form a dual pair (see [20] for details).

The main example of Poisson group actions is the so-called dressing action. The dressing action may be described as follows (see [20], [25]).

Proposition 8.2. *Let G be a connected simply connected Poisson–Lie group with factorizable tangent Lie bialgebra, G^* the dual group. Then there exists a unique right local Poisson group action*

$$G^* \times G \rightarrow G^*, ((L_+, L_-), g) \mapsto g \circ (L_+, L_-),$$

such that the identity mapping $\mu : G^ \rightarrow G^*$ is the moment map for this action.*

Moreover, let $q : G^ \rightarrow G$ be the map defined by*

$$q(L_+, L_-) = L_- L_+^{-1}.$$

Then

$$q(g \circ (L_+, L_-)) = g^{-1} L_- L_+^{-1} g.$$

The notion of Poisson group actions may be generalized as follows. Let $A \times M \rightarrow M$ be a Poisson group action of a Poisson–Lie group A on a Poisson manifold M . A subgroup $K \subset A$ is called admissible if the set $C^\infty(M)^K$ of K -invariants is a Poisson subalgebra in $C^\infty(M)$. If space M/K is a smooth manifold, we may identify the algebras $C^\infty(M/K)$ and $C^\infty(M)^K$. Hence there exists a Poisson structure on M/K such that the canonical projection $M \rightarrow M/K$ is a Poisson map.

Proposition 8.3. ([25], Theorem 6; [20], §2) *Let $(\mathfrak{a}, \mathfrak{a}^*)$ be the tangent Lie bialgebra of a Poisson–Lie group A . A connected Lie subgroup $K \subset A$ with Lie algebra $\mathfrak{k} \subset \mathfrak{a}$ is admissible if the annihilator \mathfrak{k}^\perp of \mathfrak{k} in \mathfrak{a}^* is a Lie subalgebra $\mathfrak{k}^\perp \subset \mathfrak{a}^*$.*

We shall need the following particular example of dual pairs arising from Poisson group actions.

Let $A \times M \rightarrow M$ be a right (local) Poisson group action of a Poisson–Lie group A on a manifold M . Suppose that this action possesses a moment map $\mu : M \rightarrow A^*$. Let K be an admissible subgroup in A . Denote by \mathfrak{k} the Lie algebra of K . Assume that $\mathfrak{k}^\perp \subset \mathfrak{a}^*$ is a Lie subalgebra in \mathfrak{a}^* . Suppose also that there is a splitting $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$, and that \mathfrak{t} is a Lie subalgebra in \mathfrak{a}^* . Then the linear space \mathfrak{k}^* is naturally identified with \mathfrak{t} . Assume that $A^* = K^\perp T$ as a manifold, where K^\perp, T are the Lie subgroups of A^* corresponding to the Lie subalgebras $\mathfrak{k}^\perp, \mathfrak{t} \subset \mathfrak{a}^*$, respectively. Denote by π_{K^\perp}, π_T the projections onto K^\perp and T in this decomposition. Suppose that K^\perp is a connected subgroup in A^* and that for any $k^\perp \in K^\perp$ the transformation

$$(8.3) \quad \begin{aligned} \mathfrak{t} &\rightarrow \mathfrak{t}, \\ t &\mapsto (\text{Ad}(k^\perp)t)_\mathfrak{t}, \end{aligned}$$

where the subscript \mathfrak{t} stands for the \mathfrak{t} -component with respect to the decomposition $\mathfrak{a}^* = \mathfrak{t} \oplus \mathfrak{k}^\perp$, is invertible. The following proposition is a slight generalization of Theorem 14 in [30]. The proof given in [30] still applies under the conditions imposed on K, K^\perp and T above.

Proposition 8.4. *Define a map $\bar{\mu} : M \rightarrow T$ by*

$$\bar{\mu} = \pi_T \mu.$$

Then

(i) $\bar{\mu}^*(C^\infty(T))$ *is a Poisson subalgebra in $C^\infty(M)$, and hence one can equip T with a Poisson structure such that $\bar{\mu} : M \rightarrow T$ is a Poisson map.*

(ii) *Moreover, the algebra $C^\infty(M)^K$ is the centralizer of $\bar{\mu}^*(C^\infty(T))$ in the Poisson algebra $C^\infty(M)$. In particular, if M/K is a smooth manifold the maps*

$$(8.4) \quad \begin{array}{ccc} & M & \\ \pi \swarrow & & \searrow \bar{\mu} \\ M/K & & T \end{array},$$

form a dual pair.

Remark 8.5. *Let $t \in T$ be as in Lemma 8.1. Assume that $\pi(\bar{\mu}^{-1}(t))$ is a smooth manifold (M/K does not need to be smooth). Then the algebra $C^\infty(\pi(\bar{\mu}^{-1}(t)))$ is isomorphic to the reduced Poisson algebra $C^\infty(\bar{\mu}^{-1}(t))^{C^\infty(T)}$.*

Remark 8.6. *In the proof of Theorem 14 in [30] we obtained a formula which relates the action of the Poisson algebra $\bar{\mu}^*(C^\infty(T))$ and the action of K on $C^\infty(M)$. Let $X \in \mathfrak{k}$ and \hat{X} be the corresponding vector field on M , ξ_φ the Hamiltonian vector field of $\varphi \in C^\infty(M)$. Then*

$$(8.5) \quad \begin{aligned} L_{\hat{X}}\varphi &= \langle \text{Ad}(\pi_{K^\perp}\mu)(\bar{\mu}^*\theta_T), X \rangle (\xi_\varphi) = \\ &\langle \text{Ad}(\pi_{K^\perp}\mu)(\theta_T), X \rangle (\bar{\mu}_*(\xi_\varphi)), \end{aligned}$$

where θ_T is the universal right invariant Cartan form on T .

9. QUANTIZATION OF POISSON-LIE GROUPS AND Q-W ALGEBRAS

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra. Let $s \in W$ be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and Δ_+ the system of positive roots associated to s . Observe that cocycle (4.12) equips \mathfrak{g} with the structure of a factorizable Lie bialgebra. Using the identification $\text{End } \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$ the corresponding r -matrix may be represented as

$$r^s = P_+ - P_- + \frac{1+s}{1-s} P_{\mathfrak{h}'},$$

where P_+, P_- and $P_{\mathfrak{h}'}$ are the projection operators onto $\mathfrak{n}_+, \mathfrak{n}_-$ and \mathfrak{h}' in the direct sum

$$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h}' + \mathfrak{h}'^\perp + \mathfrak{n}_-,$$

where \mathfrak{h}'^\perp is the orthogonal complement to \mathfrak{h}' in \mathfrak{h} with respect to the Killing form.

Let G be the connected simply connected simple Poisson-Lie group with the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, G^* the dual group. Note that by Proposition 17 in [30] Poisson-Lie groups G^* corresponding to different Weyl group elements $s \in W$ are isomorphic as Poisson manifolds, and as Poisson manifolds all Poisson-Lie groups G^* are isomorphic to the Poisson-Lie group G_0^* associated to the standard bialgebra structure on \mathfrak{g} with $r = P_+ - P_-$. This is the quasiclassical version of Theorem 4.1.

Observe that G is an algebraic group (see §104, Theorem 12 in [36]).

Note also that

$$r_+^s = P_+ + \frac{1}{1-s} P_{\mathfrak{h}'} + \frac{1}{2} P_{\mathfrak{h}'^\perp}, \quad r_-^s = -P_- + \frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}'^\perp},$$

and hence the subspaces \mathfrak{b}_\pm and \mathfrak{n}_\pm defined by (7.3) coincide with the Borel subalgebras in \mathfrak{g} and their nilradicals, respectively. Therefore every element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(9.1) \quad (L_+, L_-) = (h_+, h_-)(n_+, n_-),$$

where $n_\pm \in N_\pm$, $h_+ = \exp((\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}',\perp})x)$, $h_- = \exp((\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}',\perp})x)$, $x \in \mathfrak{h}$. In particular, G^* is a solvable algebraic subgroup in $G \times G$.

In terms of factorization (9.1) and a similar factorization in the Poisson–Lie group G_0^* ,

$$(L'_+, L'_-) = (h'_+, h'_-)(n'_+, n'_-), \quad n_\pm \in N_\pm, \quad h'_+ = \exp(\frac{1}{2}x'), \quad h'_- = \exp(-\frac{1}{2}x'), \quad x' \in \mathfrak{h},$$

the Poisson manifold isomorphism $G_0^* \rightarrow G^*$ established in Proposition 17 in [30] takes the form

$$L' \mapsto tL't^{-1} = L, \quad L' \in G_0^*, L \in G^*, \quad L' = L'_-(L'_+)^{-1}, L = L_-L_+^{-1}, \quad t = e^{Ax'},$$

where $A \in \text{End } \mathfrak{h}$ is an arbitrary endomorphism of \mathfrak{h} commuting with s and satisfying the equation

$$A - A^* = \frac{1}{2} \frac{1+s}{1-s} P_{\mathfrak{h}' }.$$

For every algebraic variety V we denote by $\mathbb{C}[V]$ the algebra of regular functions on V . Our main object will be the algebra of regular functions on G^* , $\mathbb{C}[G^*]$. This algebra may be explicitly described as follows. Let π_V be a finite-dimensional representation of G . Then matrix elements of $\pi_V(L_\pm)$ are well-defined functions on G^* , and $\mathbb{C}[G^*]$ is the subspace in $C^\infty(G^*)$ generated by matrix elements of $\pi_V(L_\pm)$, where V runs through all finite-dimensional representations of G .

The elements $L^{\pm,V} = \pi_V(L_\pm)$ may be viewed as elements of the space $\mathbb{C}[G^*] \otimes \text{End } V$. For every two finite-dimensional \mathfrak{g} modules V and W we denote $r_+^{s,VW} = (\pi_V \otimes \pi_W)r_+^s$, where r_+^s is regarded as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

Proposition 9.1. ([26], Section 2) $\mathbb{C}[G^*]$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$, the Poisson brackets of the elements $L^{\pm,V}$ are given by

$$(9.2) \quad \begin{aligned} \{L_1^{\pm,W}, L_2^{\pm,V}\} &= 2[r_+^{s,VW}, L_1^{\pm,W} L_2^{\pm,V}], \\ \{L_1^{-,W}, L_2^{+,V}\} &= 2[r_+^{s,VW}, L_1^{-,W} L_2^{+,V}], \end{aligned}$$

where

$$L_1^{\pm,W} = L^{\pm,W} \otimes I_V, \quad L_2^{\pm,V} = I_W \otimes L^{\pm,V},$$

and I_X is the unit matrix in X .

Moreover, the map $\Delta : \mathbb{C}[G^*] \rightarrow \mathbb{C}[G^*] \otimes \mathbb{C}[G^*]$ dual to the multiplication in G^* ,

$$(9.3) \quad \Delta(L_{ij}^{\pm,V}) = \sum_k L_{ik}^{\pm,V} \otimes L_{kj}^{\pm,V},$$

is a homomorphism of Poisson algebras, and the map $S : \mathbb{C}[G^*] \rightarrow \mathbb{C}[G^*]$,

$$S(L_{ij}^{\pm,V}) = (L_{ij}^{\pm,V})_{ij}^{-1}$$

is an antihomomorphism of Poisson algebras.

Remark 9.7. Recall that a Poisson–Hopf algebra is a Poisson algebra which is also a Hopf algebra such that the comultiplication is a homomorphism of Poisson algebras and the antipode is an antihomomorphism of Poisson algebras. According to Proposition 9.1 $\mathbb{C}[G^*]$ is a Poisson–Hopf algebra.

Now we construct a quantization of the Poisson–Hopf algebra $\mathbb{C}[G^*]$. For technical reasons we shall need an extension of the algebra $U_{\mathcal{A}}^s(\mathfrak{g})$ to an algebra $U_{\mathcal{A}'}^s(\mathfrak{g}) = U_{\mathcal{A}}^s(\mathfrak{g}) \otimes_{\mathcal{A}} \mathcal{A}'$, where $\mathcal{A}' = \mathbb{C}[q^{\frac{1}{2d}}, q^{-\frac{1}{2d}}, \frac{1}{[2]_{q_i}}, \dots, \frac{1}{[r]_{q_i}}, \frac{1-q^{\frac{1}{2d}}}{1-q_i^{-2}}]_{i=1, \dots, l}$. Note that the ratios $\frac{1-q^{\frac{1}{2d}}}{1-q_i^{-2}}$ have no singularities when $q = 1$, and we can define a localization, $\mathcal{A}'/(1-q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$ as well as similar localizations for other generic values of ε , $\mathcal{A}'/(\varepsilon^{\frac{1}{2d}} - q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$ and the corresponding localizations of algebras over \mathcal{A}' . $U_{\mathcal{A}'}^s(\mathfrak{g})$ is naturally a Hopf algebra with the comultiplication and the antipode induced from $U_{\mathcal{A}}^s(\mathfrak{g})$.

First, using arguments similar to those applied in the end of Section 6 where we defined the action of the element \mathcal{R}^s in tensor products of finite–dimensional representations, one can show that for any finite–dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$ module V the invertible elements ${}^qL^{\pm, V}$ given by

$${}^qL^{+, V} = (id \otimes \pi_V) \mathcal{R}_{21}^{s-1} = (id \otimes \pi_V S^s) \mathcal{R}_{21}^s, \quad {}^qL^{-, V} = (id \otimes \pi_V) \mathcal{R}^s.$$

are well–defined elements of $U_{\mathcal{A}}^s(\mathfrak{g}) \otimes \text{End} V$ (compare with [11]). If we fix a basis in V , ${}^qL^{\pm, V}$ may be regarded as matrices with matrix elements $({}^qL^{\pm, V})_{ij}$ being elements of $U_{\mathcal{A}}^s(\mathfrak{g})$. We also recall that one can define an operator $R^{VW} = (\pi_V \otimes \pi_W) \mathcal{R}^s$ (see Section 6).

From the Yang–Baxter equation for \mathcal{R} we get relations between ${}^qL^{\pm, V}$:

$$(9.4) \quad R^{VW} {}^qL_1^{\pm, W} {}^qL_2^{\pm, V} = {}^qL_2^{\pm, V} {}^qL_1^{\pm, W} R^{VW},$$

$$(9.5) \quad R^{VW} {}^qL_1^{-, W} {}^qL_2^{+, V} = {}^qL_2^{+, V} {}^qL_1^{-, W} R^{VW}.$$

By ${}^qL_1^{\pm, W}$, ${}^qL_2^{\pm, V}$ we understand the following matrices in $V \otimes W$ with entries being elements of $U_{\mathcal{A}}^s(\mathfrak{g})$:

$${}^qL_1^{\pm, W} = {}^qL^{\pm, W} \otimes I_V, \quad {}^qL_2^{\pm, V} = I_W \otimes {}^qL^{\pm, V},$$

where I_X is the unit matrix in X .

From (3.7) we can obtain the action of the comultiplication on the matrices ${}^qL^{\pm, V}$:

$$(9.6) \quad \Delta_s({}^qL_{ij}^{\pm, V}) = \sum_k {}^qL_{ik}^{\pm, V} \otimes {}^qL_{kj}^{\pm, V}$$

and the antipode,

$$(9.7) \quad S_s({}^qL_{ij}^{\pm, V}) = ({}^qL_{ij}^{\pm, V})_{ij}^{-1}.$$

We denote by $\mathbb{C}_{\mathcal{A}'}[G^*]$ the Hopf subalgebra in $U_{\mathcal{A}'}^s(\mathfrak{g})$ generated by matrix elements of $({}^qL^{\pm, V})^{\pm 1}$, where V runs through all finite–dimensional representations of $U_{\mathcal{A}}^s(\mathfrak{g})$.

Since $\mathcal{R}^s = 1 \otimes 1 \pmod{h}$ relations (9.4) and (9.5) imply that the quotient algebra $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*]$ is commutative, and one can equip it with a Poisson structure given by

$$(9.8) \quad \{x_1, x_2\} = \frac{1}{2d} \frac{[a_1, a_2]}{q^{\frac{1}{2d}} - 1} \pmod{(q^{\frac{1}{2d}} - 1)},$$

where $a_1, a_2 \in \mathbb{C}_{\mathcal{A}'}[G^*]$ reduce to $x_1, x_2 \in \mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*] \pmod{(q^{\frac{1}{2d}} - 1)}$. Obviously, the maps (9.6) and (9.7) induce a comultiplication and an antipode on $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*]$ compatible with the introduced Poisson structure, and the quotient $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*]$ becomes a Poisson–Hopf algebra.

Proposition 9.2. ([32], **Proposition 10.2**) *The Poisson–Hopf algebra $\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*]$ is isomorphic to $\mathbb{C}[G^*]$ as a Poisson–Hopf algebra.*

We shall call the map $p : \mathbb{C}_{\mathcal{A}'}[G^*] \rightarrow \mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*] = \mathbb{C}[G^*]$ the quasiclassical limit.

From the definition of the elements ${}^qL^{\pm, V}$ it follows that $\mathbb{C}_{\mathcal{A}'}[G^*]$ is the subalgebra in $U_{\mathcal{A}'}^s(\mathfrak{g})$ generated by the elements $\prod_{j=1}^l t_j^{\pm 2dp_{ij}}$, $\prod_{j=1}^l t_j^{\pm 2dp_{ji}}$, $i = 1, \dots, l$, $\tilde{e}_{\beta} = (1 - q_{\beta}^{-2})e_{\beta}$, $\tilde{f}_{\beta} = (1 - q_{\beta}^{-2})e^{h\beta^{\vee}}f_{\beta}$, $\beta \in \Delta_+$.

Now using the Hopf algebra $\mathbb{C}_{\mathcal{A}'}[G^*]$ we shall define quantum versions of W-algebras. From the definition of the elements ${}^qL^{\pm,V}$ it follows that the matrix elements of ${}^qL^{\pm,V^{\pm 1}}$ form Hopf subalgebras $\mathbb{C}_{\mathcal{A}'}[B_{\pm}] \subset \mathbb{C}_{\mathcal{A}'}[G^*]$, and that $\mathbb{C}_{\mathcal{A}'}[G^*]$ contains the subalgebra $\mathbb{C}_{\mathcal{A}'}[N_-]$ generated by elements $\tilde{e}_{\beta} = (1 - q_{\beta}^{-2})e_{\beta}$, $\beta \in \Delta_+$.

Suppose that the ordering of the root system Δ_+ is fixed as in formula (5.8). Denote by $\mathbb{C}_{\mathcal{A}'}[M_-]$ the subalgebra in $\mathbb{C}_{\mathcal{A}'}[N_-]$ generated by elements \tilde{e}_{β} , $\beta \in \Delta_{\mathfrak{m}_+}$.

By construction $\mathbb{C}_{\mathcal{A}'}[N_-]$ is a quantization of the algebra of regular functions on the algebraic subgroup $N_- \subset G^*$ corresponding to the Lie subalgebra $\mathfrak{n}_- \subset \mathfrak{g}^*$, and $\mathbb{C}_{\mathcal{A}'}[M_-]$ is a quantization of the algebra of regular functions on the algebraic subgroup $M_- \subset G^*$ corresponding to the Lie subalgebra $\mathfrak{m}_- \subset \mathfrak{g}^*$ in the sense that $p(\mathbb{C}_{\mathcal{A}'}[N_-]) = \mathbb{C}[N_-]$ and $p(\mathbb{C}_{\mathcal{A}'}[M_-]) = \mathbb{C}[M_-]$. We also denote by M_+ the algebraic subgroup $M_+ \subset G^*$ corresponding to the Lie subalgebra $\mathfrak{m}_+ \subset \mathfrak{g}^*$.

We claim that the defining relations in the subalgebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ are given by a formula similar to (6.4). Indeed, consider the defining relations in the subalgebra $U_{\mathcal{A}}^s(\mathfrak{m}_+)$,

$$e_{\alpha}e_{\beta} - q^{(\alpha,\beta) + (\frac{1+s}{1-s}P_{\mathfrak{h}'}\alpha,\beta)}e_{\beta}e_{\alpha} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C'(k_1, \dots, k_n) e_{\delta_1}^{k_1} e_{\delta_2}^{k_2} \dots e_{\delta_n}^{k_n},$$

where $C'(k_1, \dots, k_n) \in \mathcal{A}$. Commutation relations between quantum analogues of root vectors obtained in Proposition 4.2 in [17] imply that each function $C'(k_1, \dots, k_n)$ has a zero of order $k_1 + \dots + k_n - 1$ at point $q = 1$. Therefore one can write the following defining relations for the generators $\tilde{e}_{\beta} = (1 - q_{\beta}^{-2})e_{\beta}$, $\beta \in \Delta_{\mathfrak{m}_+}$ in the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$,

$$(9.9) \quad \tilde{e}_{\alpha}\tilde{e}_{\beta} - q^{(\alpha,\beta) + (\frac{1+s}{1-s}P_{\mathfrak{h}'}\alpha,\beta)}\tilde{e}_{\beta}\tilde{e}_{\alpha} = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C''(k_1, \dots, k_n) \tilde{e}_{\delta_1}^{k_1} \tilde{e}_{\delta_2}^{k_2} \dots \tilde{e}_{\delta_n}^{k_n},$$

where $C''(k_1, \dots, k_n) \in \mathcal{A}'$, and each function $C''(k_1, \dots, k_n)$ has a zero of order 1 at point $q = 1$.

Now arguments similar to those used in the proof of Theorem 5.3 show that the map $\chi_q^s : \mathbb{C}_{\mathcal{A}'}[M_-] \rightarrow \mathcal{A}'$,

$$(9.10) \quad \chi_q^s(\tilde{e}_{\beta}) = \begin{cases} 0 & \beta \notin \{\gamma_1, \dots, \gamma_{l'}\} \\ k_i & \beta = \gamma_i, k_i \in \mathcal{A}' \end{cases},$$

is a character of $\mathbb{C}_{\mathcal{A}'}[M_-]$. Denote by $\mathbb{C}_{\chi_q^s}$ the rank one representation of the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ defined by the character χ_q^s .

For any finite-dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$ -module V let ${}^qL^V = {}^qL^{-,V}{}^qL^{+,V^{-1}} = (id \otimes \pi_V)\mathcal{R}^s\mathcal{R}_{21}^s$. Let $\mathbb{C}_{\mathcal{A}'}[G_*]$ be the \mathcal{A}' -subalgebra in $\mathbb{C}_{\mathcal{A}'}[G^*]$ generated by the matrix entries of ${}^qL^V$, where V runs over all finite-dimensional representations of $U_{\mathcal{A}}^s(\mathfrak{g})$. From the commutation relations for elements ${}^qL^{\pm,V}$ it follows that ${}^qL^V$ satisfy the following relations:

$$R^{VW}{}^qL_1^W R^{WV}{}^qL_2^V = {}^qL_2^V R^{VW}{}^qL_1^W R^{WV}.$$

Define the right adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{g})$ on $U_{\mathcal{A}'}^s(\mathfrak{g})$ by the formula

$$(9.11) \quad \text{Ad}x(w) = S_s^{-1}(x_2)wx_1,$$

where we use the abbreviated notation for the coproduct $\Delta_s(x) = x_1 \otimes x_2$, $x \in U_{\mathcal{A}'}^s(\mathfrak{g})$, $w \in U_{\mathcal{A}'}^s(\mathfrak{g})$.

Note that by Lemma 2.2 in [14]

$$(9.12) \quad \text{Ad}x(wz) = \text{Ad}x_2(w)\text{Ad}x_1(z).$$

Observe also that by definition the adjoint action introduced above is dual to a restriction of the dressing coaction of the quantization of the algebra of regular functions on the Poisson-Lie group G on the space $\mathbb{C}_{\mathcal{A}'}[G^*]$. Therefore the subspace $\mathbb{C}_{\mathcal{A}'}[G^*] \subset U_{\mathcal{A}'}^s(\mathfrak{g})$ is stable under the adjoint action. The subalgebra $\mathbb{C}_{\mathcal{A}'}[G_*] \subset \mathbb{C}_{\mathcal{A}'}[G^*]$ is also stable under the dressing coaction (see [26], Section 3), and hence $\mathbb{C}_{\mathcal{A}'}[G_*]$ is stable under the adjoint action.

Proposition 9.3. *Let $\varepsilon \in \mathbb{C}$ be generic such that $[r]_{\varepsilon_i}! \neq 0$, $\varepsilon \neq 0$, $i = 1, \dots, l$. Define the complex associative algebra $\mathbb{C}_\varepsilon[G_*] = \mathbb{C}_{\mathcal{A}'}[G_*]/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})\mathbb{C}_{\mathcal{A}'}[G_*]$. Then the algebra $\mathbb{C}_\varepsilon[G_*]$ can be identified with the Ad locally finite part $U_\varepsilon^s(\mathfrak{g})^{fin}$ of $U_\varepsilon^s(\mathfrak{g})$,*

$$U_\varepsilon^s(\mathfrak{g})^{fin} = \{x \in U_\varepsilon^s(\mathfrak{g}) : \dim(\text{Ad}U_\varepsilon^s(\mathfrak{g})(x)) < +\infty\},$$

where the adjoint action of the algebra $U_\varepsilon^s(\mathfrak{g})$ on itself is defined by formula (9.11).

Proof. Indeed, let V_i , $i = 1, \dots, l$ be the fundamental representations of $U_{\mathcal{A}}^s(\mathfrak{g})$ with highest weights Y_i , $i = 1, \dots, l$. From formula (4.10) and from the definition of ${}^qL^V = (id \otimes \pi_V)\mathcal{R}^s\mathcal{R}_{21}^s$ it follows that the matrix element $(id \otimes v_i^*)\mathcal{R}^s\mathcal{R}_{21}^s(id \otimes v_i)$ of ${}^qL^{V_i}$ corresponding to the highest weight vector v_i of V_i and to the lowest weight vector $v_i^* \in V_i^*$ of the dual representation V_i^* , normalized in such a way that $v_i^*(v_i) = 1$, coincides with L_i^2 . This implies that L_i^2 are elements of the algebra $\mathbb{C}_\varepsilon[G_*] \subset U_\varepsilon^s(\mathfrak{g})$ as well. Denote by $\mathfrak{H} \subset \mathbb{C}_\varepsilon[G_*] \subset U_\varepsilon^s(\mathfrak{g})$ the subalgebra generated by the elements $L_i^2 \in \mathbb{C}_\varepsilon[G_*]$, $i = 1, \dots, l$. By Theorem 7.1.6 and Lemma 7.1.16 in [13] $U_\varepsilon^s(\mathfrak{g})^{fin} = \text{Ad}U_\varepsilon^s(\mathfrak{g})\mathfrak{H}$. Since $\mathbb{C}_\varepsilon[G_*]$ is stable under the adjoint action we have an inclusion, $U_\varepsilon^s(\mathfrak{g})^{fin} \subset \mathbb{C}_\varepsilon[G_*]$. On the other hand from formula (3.26) in [26] it follows that the dressing coaction on $\mathbb{C}_\varepsilon[G_*]$ is locally cofinite, and hence the adjoint action of $U_\varepsilon^s(\mathfrak{g})$ on $\mathbb{C}_\varepsilon[G_*]$ is locally finite. Hence $\mathbb{C}_\varepsilon[G_*] \subset U_\varepsilon^s(\mathfrak{g})^{fin}$, and $\mathbb{C}_\varepsilon[G_*] = U_\varepsilon^s(\mathfrak{g})^{fin}$. \square

Denote by I_q the left ideal in $\mathbb{C}_{\mathcal{A}'}[G^*]$ generated by the kernel of χ_q^s , and by $\rho_{\chi_q^s}$ the canonical projection $\mathbb{C}_{\mathcal{A}'}[G^*] \rightarrow \mathbb{C}_{\mathcal{A}'}[G^*]/I_q$. Let $Q_{\mathcal{A}'}$ be the image of $\mathbb{C}_{\mathcal{A}'}[G^*]$ under the projection $\rho_{\chi_q^s}$.

Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or that the set $\gamma_1, \dots, \gamma_n$ is empty, and hence the segment Δ_{s^1} is of the form $\Delta_{s^1} = \{\gamma_1, \dots, \gamma_n\}$. Then from formula (4.13) it follows that $\Delta_s(U_{\mathcal{A}'}^s(\mathfrak{m}_+)) \subset U_{\mathcal{A}'}^s(\mathfrak{m}_+) \otimes U_{\mathcal{A}'}^s(\mathfrak{b}_+)$, where $U_{\mathcal{A}'}^s(\mathfrak{m}_+) = U_{\mathcal{A}}^s(\mathfrak{m}_+) \otimes_{\mathcal{A}} \mathcal{A}'$, $U_{\mathcal{A}'}^s(\mathfrak{b}_+) = U_{\mathcal{A}}^s(\mathfrak{b}_+) \otimes_{\mathcal{A}} \mathcal{A}'$.

Now observe that from the definition of the normal ordering (5.8) it follows that the r.h.s. in formula (9.9) belongs to the subspace $(1 - q_\alpha^{-2})\text{Ker}\chi_q^s$ and hence dividing (9.9) by $(1 - q_\alpha^{-2})$ we obtain

$$(9.13) \quad e_\alpha \tilde{e}_\beta - q^{(\alpha, \beta) + (\frac{1+s}{1-s}P_{\mathfrak{b}'} + \alpha, \beta)} \tilde{e}_\beta e_\alpha = \sum_{\alpha < \delta_1 < \dots < \delta_n < \beta} C'''(k_1, \dots, k_n) \tilde{e}_{\delta_1}^{k_1} \tilde{e}_{\delta_2}^{k_2} \dots \tilde{e}_{\delta_n}^{k_n},$$

where $C'''(k_1, \dots, k_n) = C''(k_1, \dots, k_n)/(1 - q_\alpha^{-2}) \in \mathcal{A}'$.

Therefore we have the inclusion $[U_{\mathcal{A}'}^s(\mathfrak{m}_+), \text{Ker}\chi_q^s] \subset \text{Ker}\chi_q^s$. Using this inclusion, formula (9.11), the fact that $\Delta_s(U_{\mathcal{A}'}^s(\mathfrak{m}_+)) \subset U_{\mathcal{A}'}^s(\mathfrak{m}_+) \otimes U_{\mathcal{A}'}^s(\mathfrak{b}_+)$ (see formula (4.13)) we deduce that the adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $\mathbb{C}_{\mathcal{A}'}[G^*]$ induces an adjoint action on $Q_{\mathcal{A}'}$ which we also call the adjoint action and denote it by Ad.

Let $\mathbb{C}_{\mathcal{A}'}$ be the trivial representation of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ given by the counit. Consider the space $W_q^s(G)$ of Ad-invariants in $Q_{\mathcal{A}'}$,

$$(9.14) \quad W_q^s(G) = \text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathbb{C}_{\mathcal{A}'}, Q_{\mathcal{A}'}).$$

Proposition 9.4. *$W_q^s(G)$ is isomorphic to the subspace of all $v + I_q \in Q_{\chi_q^s}$ such that $mv \in I_q$ (or $[m, v] \in I_q$) in $\mathbb{C}_{\mathcal{A}'}[G^*]$ for any $m \in I_q$, where $v \in \mathbb{C}_{\mathcal{A}'}[G^*]$ is any representative of $v + I_q \in Q_{\chi_q^s}$.*

Multiplication in $\mathbb{C}_{\mathcal{A}'}[G^]$ induces a multiplication on the space $W_q^s(G)$.*

Proof. From formulas (4.13) and (9.13) it follows that for $\beta_k \in \Delta_{\mathfrak{m}_+}$

$$(9.15) \quad \Delta_s(e_{\beta_k}) = e_{\beta_k} \otimes e^{h(\frac{2}{1-s}P_{\mathfrak{b}'} + P_{\mathfrak{b}'^\perp})\beta_k^\vee} + 1 \otimes e_{\beta_k} + x, \quad x \in I_{<\beta_k} \otimes U_{\mathcal{A}'}^s(\mathfrak{b}_+),$$

where $I_{<\beta_k}$ is the intersection of the ideal I_q and of the subalgebra in $\mathbb{C}_{\mathcal{A}'}[M_-]$ generated by \tilde{e}_{β_r} , $0 < \beta_r < \beta_k$.

Recall that S_s^{-1} is the antipode for the comultiplication Δ_s^{opp} and hence from the definition of the antipode, the fact that $S_s^{-1}(U_{\mathcal{A}'}^s(\mathfrak{b}_+)) \subset U_{\mathcal{A}'}^s(\mathfrak{b}_+)$ and inclusion (9.15) we must have

$$S_s^{-1}(e_{\beta_k}) + e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'} + P_{\mathfrak{b}'^\perp})\beta_k^\vee} e_{\beta_k} \in U_{\mathcal{A}'}^s(\mathfrak{b}_+)I_{<\beta_k}.$$

Therefore

$$S_s^{-1}(e_{\beta_k}) = -e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta^\vee} e_{\beta_k} + y, y \in U_{\mathcal{A}'}^s(\mathfrak{b}_+)I_{<\beta_k}$$

Now the last formula, formula (9.15) and the definition of the adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $Q'_{\mathcal{A}}$ imply that for any representative $v \in \mathbb{C}_{\mathcal{A}'}[G^*]$ of any element $v + I_q \in Q_{\mathcal{A}'}$ we have the following identity in $\mathbb{C}_{\mathcal{A}'}[G^*]$

$$(9.16) \quad \text{Ade}_{\beta_k} v = -\frac{1}{(1-q_{\beta_k}^{-2})} e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta^\vee} (\tilde{e}_{\beta_k} - \chi_q^s(\tilde{e}_{\beta_k}))v + yv + z, y \in U_{\mathcal{A}'}^s(\mathfrak{b}_+)I_{<\beta_k}, z \in I_q.$$

From the last identity it obviously follows that if $mv \in I_q$ in $\mathbb{C}_{\mathcal{A}'}[G^*]$ for any $m \in I_q$ then $v + I_q$ is invariant with respect to the adjoint action.

Now let $v + I_q \in Q_{\mathcal{A}'}$, be an element which is invariant with respect to the adjoint action,

$$\text{Ad}x(v) = \varepsilon(x)v + z', x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+), z' \in I_q.$$

Since $\beta_1 \in \Delta_{\mathfrak{m}_+}$ is the first positive root in the normal ordering (5.8) we have $I_{<\beta_1} = 0$, and (9.16) implies that

$$z' = \text{Ade}_{\beta_1} v = -\frac{1}{(1-q_{\beta_1}^{-2})} e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta_1^\vee} (\tilde{e}_{\beta_1} - \chi_q^s(\tilde{e}_{\beta_1}))v + z, z \in I_q.$$

We obtain from the last identity that

$$-\frac{1}{(1-q_{\beta_1}^{-2})} e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta_1^\vee} (\tilde{e}_{\beta_1} - \chi_q^s(\tilde{e}_{\beta_1}))v \in I_q$$

which is obviously possible only in case if $(\tilde{e}_{\beta_1} - \chi_q^s(\tilde{e}_{\beta_1}))v \in I_q$, i.e. when

$$(\tilde{e}_{\beta_1} - \chi_q^s(\tilde{e}_{\beta_1}))v \in I_q.$$

Now we proceed by induction. Assume that

$$(\tilde{e}_{\beta_r} - \chi_q^s(\tilde{e}_{\beta_r}))v \in I_q$$

for $0 < \beta_r < \beta_k$. From (9.16) we have

$$\text{Ade}_{\beta_k} v = -\frac{1}{(1-q_{\beta_k}^{-2})} e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta^\vee} (\tilde{e}_{\beta_k} - \chi_q^s(\tilde{e}_{\beta_k}))v + yv, y \in U_{\mathcal{A}'}^s(\mathfrak{b}_+)I_{<\beta_k} \in I_q.$$

By the induction hypothesis $I_{<\beta_k} v \in I_q$, and hence

$$\text{Ade}_{\beta_k} v = -\frac{1}{(1-q_{\beta_k}^{-2})} e^{-h(\frac{2}{1-s}P_{\mathfrak{b}'}+P_{\mathfrak{b}'\perp})\beta^\vee} (\tilde{e}_{\beta_k} - \chi_q^s(\tilde{e}_{\beta_k}))v \in I_q.$$

Finally an argument similar to that applied in case $k = 1$ shows that $(\tilde{e}_{\beta_k} - \chi_q^s(\tilde{e}_{\beta_k}))v \in I_q$. This establishes the induction step and proves that

$$(9.17) \quad (\tilde{e}_{\beta_r} - \chi_q^s(\tilde{e}_{\beta_r}))v \in I_q$$

for any $\beta \in \Delta_{\mathfrak{m}_+}$. Since as a left ideal I_q is generated by the elements $\tilde{e}_{\beta_r} - \chi_q^s(\tilde{e}_{\beta_r})$, $\beta \in \Delta_{\mathfrak{m}_+}$ (9.17) proves that $mv \in I_q$ in $\mathbb{C}_{\mathcal{A}'}[G^*]$ for any $m \in I_q$.

Now if $v_1, v_2 \in \mathbb{C}_{\mathcal{A}'}[G^*]$ are any representatives of elements $v_1 + I_q, v_2 + I_q \in W_q^s(G)$ the formula

$$(v_1 + I_q)(v_2 + I_q) = v_1 v_2 + I_q$$

defines a multiplication in $W_q^s(G)$. □

We call the space $W_q^s(G)$ equipped with the multiplication defined in the previous proposition the q-W algebra associated to (the conjugacy class of) the Weyl group element $s \in W$.

Now consider the Lie algebra $\mathfrak{L}_{\mathcal{A}'}$ associated to the associative algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$, i.e. $\mathfrak{L}_{\mathcal{A}'}$ is the Lie algebra which is isomorphic to $\mathbb{C}_{\mathcal{A}'}[M_-]$ as a linear space, and the Lie bracket in $\mathfrak{L}_{\mathcal{A}'}$ is given by the usual commutator of elements in $\mathbb{C}_{\mathcal{A}'}[M_-]$.

Define an action of the Lie algebra $\mathfrak{L}_{\mathcal{A}'}$ on the space $\mathbb{C}_{\mathcal{A}'}[G^*]/I_q$:

$$(9.18) \quad m \cdot (x + I_q) = \rho_{\chi_q^s}([m, x]).$$

where $x \in \mathbb{C}_{\mathcal{A}'}[G^*]$ is any representative of $x + I_q \in \mathbb{C}_{\mathcal{A}'}[G^*]/I_q$ and $m \in \mathbb{C}_{\mathcal{A}'}[M_-]$. The algebra $W_q^s(G)$ can be regarded as the intersection of the space of invariants with respect to action (9.18) with the subspace $Q_{\mathcal{A}'} \subset \mathbb{C}_{\mathcal{A}'}[G^*]/I_q$.

Note also that since χ_q^s is a character of $\mathbb{C}_{\mathcal{A}'}[M_-]$ the ideal I_q is stable under that action of $\mathbb{C}_{\mathcal{A}'}[M_-]$ on $\mathbb{C}_{\mathcal{A}'}[G^*]$ by commutators.

Denote by $\mathbb{C}_{\chi_q^s}$ the rank one representation of the algebra $\mathbb{C}_{\mathcal{A}'}[M_-]$ defined by the character χ_q^s . Using the description of the algebra $W_q^s(G)$ in terms of action (9.18) and the isomorphism $\mathbb{C}_{\mathcal{A}'}[G^*]/I_q = \mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}$ one can also define the algebra $W_q^s(G)$ as the intersection

$$W_q^s(G) = \text{Hom}_{\mathbb{C}_{\mathcal{A}'}[M_-]}(\mathbb{C}_{\chi_q^s}, \mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}) \cap Q_{\mathcal{A}'}.$$

Using Frobenius reciprocity we also have

$$\text{Hom}_{\mathbb{C}_{\mathcal{A}'}[M_-]}(\mathbb{C}_{\chi_q^s}, \mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}) = \text{End}_{\mathbb{C}_{\mathcal{A}'}[G^*]}(\mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}).$$

Hence the algebra $W_q^s(G)$ acts on the space $\mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}$ from the right by operators commuting with the natural left $\mathbb{C}_{\mathcal{A}'}[G^*]$ -action on $\mathbb{C}_{\mathcal{A}'}[G^*] \otimes_{\mathbb{C}_{\mathcal{A}'}[M_-]} \mathbb{C}_{\chi_q^s}$. By the definition of $W_q^s(G)$ this action preserves $Q_{\mathcal{A}'}$ and by the above presented arguments it commutes with the natural left $\mathbb{C}_{\mathcal{A}'}[G^*]$ -action on $Q_{\mathcal{A}'}$.

Thus $Q_{\mathcal{A}'}$ is a $\mathbb{C}_{\mathcal{A}'}[G^*]$ - $W_q^s(G)$ bimodule equipped also with the adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$. By (9.12) the adjoint action satisfies

$$(9.19) \quad \text{Ad}_s(yv) = \text{Ad}_{x_2}(y)\text{Ad}_{x_1}(v), \quad x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+), \quad y \in \mathbb{C}_{\mathcal{A}'}[G_*], \quad v \in Q_{\mathcal{A}'},$$

and $\Delta_s(x) = x_1 \otimes x_2$.

Denote by v_0 the image of the element $1 \in \mathbb{C}_{\mathcal{A}'}[G_*]$ in the quotient $Q_{\mathcal{A}'}$ under the canonical projection $\mathbb{C}_{\mathcal{A}'}[G_*] \rightarrow Q_{\mathcal{A}'}$. Obviously v_0 is the generating vector for $Q_{\mathcal{A}'}$ as a module over $\mathbb{C}_{\mathcal{A}'}[G_*]$. Using formula (9.19) and recalling that $Q_{\mathcal{A}'}$ is a $\mathbb{C}_{\mathcal{A}'}[G_*]$ - $W_q^s(G)$ bimodule, for $x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+)$, $y \in \mathbb{C}_{\mathcal{A}'}[G_*]$, and for a representative $w \in \mathbb{C}_{\varepsilon}[G_*]$ of an element $w + I_q \in W_q^s(G)$ we have

$$\begin{aligned} \text{Ad}_s(wyv_0) &= \text{Ad}_s(ywv_0) = \text{Ad}_{x_2}(y)\text{Ad}_{x_1}(wv_0) = \\ &= \text{Ad}_{x_2}(y)\varepsilon(x_1)wv_0 = \text{Ad}_s(y)wv_0 = w\text{Ad}_s(yv_0). \end{aligned}$$

Since $Q_{\mathcal{A}'}$ is generated by the vector v_0 over $\mathbb{C}_{\mathcal{A}'}[G_*]$ the last relation implies that the action of $W_q^s(G)$ on $Q_{\mathcal{A}'}$ commutes with the adjoint action.

We can summarize the results of the above discussion in the following proposition.

Proposition 9.5. *The space $Q_{\mathcal{A}'}$ is naturally equipped with the structure of a left $\mathbb{C}_{\mathcal{A}'}[G_*]$ -module, a right $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module via the adjoint action and a right $W_q^s(G)$ -module in such a way that the left $\mathbb{C}_{\mathcal{A}'}[G_*]$ -action and the right $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -action commute with the right $W_q^s(G)$ -action and compatibility condition (9.19) is satisfied.*

In conclusion we remark that by specializing q to a particular value $\varepsilon \in \mathbb{C}$ such that $[r]_{\varepsilon^i}! \neq 0$, $\varepsilon^{d_i} \neq 0$, $i = 1, \dots, l$, one can define a complex associative algebra $\mathbb{C}_{\varepsilon}[G_*] = \mathbb{C}_{\mathcal{A}'}[G_*]/(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})\mathbb{C}_{\mathcal{A}'}[G_*]$, its subalgebra $\mathbb{C}_{\varepsilon}[M_-]$ with a nontrivial character χ_{ε}^s and the corresponding W-algebra

$$(9.20) \quad W_{\varepsilon}^s(G) = \text{Hom}_{U_{\varepsilon}^s(\mathfrak{m}_+)}(\mathbb{C}_{\varepsilon}, Q_{\varepsilon}),$$

where \mathbb{C}_ε is the trivial representation of the algebra $U_\varepsilon^s(\mathfrak{m}_+)$ induced by the counit, $Q_\varepsilon = Q_{\mathcal{A}'} / Q_{\mathcal{A}'}(q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}})$.

Obviously, for generic ε we have $W_\varepsilon^s(G) = W_q^s(G) / (q^{\frac{1}{2d}} - \varepsilon^{\frac{1}{2d}}) W_q^s(G)$.

10. POISSON REDUCTION AND Q-W ALGEBRAS

In this section we shall analyze the quasiclassical limit of the algebra $W_q^s(G)$. Using results of Section 8 we realize this limit algebra as the algebra of functions on a reduced Poisson manifold. We assume again that the roots $\gamma_1, \dots, \gamma_n$ are simple or that the set $\gamma_1, \dots, \gamma_n$ is empty.

Denote by χ^s the character of the Poisson subalgebra $\mathbb{C}[M_-]$ such that $\chi^s(p(x)) = \chi_q^s(x) \pmod{(q^{\frac{1}{2d}} - 1)}$ for every $x \in \mathbb{C}_{\mathcal{A}'}[M_-]$.

Note that under the projection $p : \mathbb{C}_{\mathcal{A}'}[G^*] \rightarrow \mathbb{C}_{\mathcal{A}'}[G^*] / (1 - q^{\frac{1}{2d}}) \mathbb{C}_{\mathcal{A}'}[G^*]$ and the canonical projection $U_{\mathcal{A}'}^s(\mathfrak{m}_+) \rightarrow U_{\mathcal{A}'}^s(\mathfrak{m}_+) / U_{\mathcal{A}'}^s(\mathfrak{m}_+)(1 - q^{\frac{1}{2d}}) = U(\mathfrak{m}_+)$ the right adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $\mathbb{C}_{\mathcal{A}'}[G^*]$ induces the right infinitesimal dressing action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[G^*]$, and the image of the algebra $\mathbb{C}_{\mathcal{A}'}[G^*]$ under the projection p is a certain subalgebra of $\mathbb{C}[G^*]$ that we denote by $\mathbb{C}[G_*]$. Let $I = p(I_q)$ be the ideal in $\mathbb{C}[G^*]$ generated by the kernel of χ^s . Then by the discussion after formula (9.18) the Poisson algebra $W^s(G) = W_q^s(G) / (q^{\frac{1}{2d}} - 1) W_q^s(G)$ is the subspace of all $x + I \in Q_1$, $Q_1 = Q_{\mathcal{A}'} / (1 - q^{\frac{1}{2d}}) Q_{\mathcal{A}'} \subset \mathbb{C}[G^*] / I$, such that $\{m, x\} \in I$ for any $m \in \mathbb{C}[M_-]$, and the Poisson bracket in $W^s(G)$ takes the form $\{(x + I), (y + I)\} = \{x, y\} + I$, $x + I, y + I \in W^s(G)$. We shall also write $W^s(G) = (\mathbb{C}[G^*] / I)^{\mathbb{C}[M_-]} \cap Q_1$, where the action of the Poisson algebra $\mathbb{C}[M_-]$ on the space $\mathbb{C}[G^*] / I$ is defined as follows

$$(10.1) \quad x \cdot (v + I) = \rho_{\chi^s}(\{x, v\}),$$

$v \in \mathbb{C}[G^*]$ is any representative of $v + I \in \mathbb{C}[G^*] / I$ and $x \in \mathbb{C}[M_-]$.

We shall describe the space of invariants $(\mathbb{C}[G^*] / I)^{\mathbb{C}[M_-]}$ with respect to this action by analyzing “dual geometric objects”. First observe that algebra $(\mathbb{C}[G^*] / I)^{\mathbb{C}[M_-]}$ is a particular example of the reduced Poisson algebra introduced in Lemma 8.1.

Indeed, recall that according to (9.1) any element $(L_+, L_-) \in G^*$ may be uniquely written as

$$(10.2) \quad (L_+, L_-) = (h_+, h_-)(n_+, n_-),$$

where $n_\pm \in N_\pm$, $h_+ = \exp((\frac{1}{1-s}P_{\mathfrak{h}'} + \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $h_- = \exp((\frac{s}{1-s}P_{\mathfrak{h}'} - \frac{1}{2}P_{\mathfrak{h}'^\perp})x)$, $x \in \mathfrak{h}$.

Formula (9.1) and decomposition of N_- into products of one-dimensional subgroups corresponding to roots also imply that every element L_- may be represented in the form

$$(10.3) \quad L_- = \exp \left[\sum_{i=1}^l b_i \left(\frac{s}{1-s} P_{\mathfrak{h}'} - \frac{1}{2} P_{\mathfrak{h}'^\perp} \right) H_i \right] \times \prod_{\beta} \exp[b_\beta X_{-\beta}], \quad b_i, b_\beta \in \mathbb{C},$$

where the product over roots is taken in the same order as in normal ordering (5.8).

Now define a map $\mu_{M_+} : G^* \rightarrow M_-$ by

$$(10.4) \quad \mu_{M_+}(L_+, L_-) = m_-,$$

where for L_- given by (10.3) m_- is defined as follows

$$m_- = \prod_{\beta \in \Delta_{\mathfrak{m}_+}} \exp[b_\beta X_{-\beta}],$$

and the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$.

By definition μ_{M_+} is a morphism of algebraic varieties. We also note that by definition $\mathbb{C}[M_-] = \{\varphi \in \mathbb{C}[G^*] : \varphi = \varphi(m_-)\}$. Therefore $\mathbb{C}[M_-]$ is generated by the pullbacks of regular functions on M_- with respect to the map μ_{M_+} . Since $\mathbb{C}[M_-]$ is a Poisson subalgebra in $\mathbb{C}[G^*]$, and regular

functions on M_- are dense in $C^\infty(M_-)$ on every compact subset, we can equip the manifold M_- with the Poisson structure in such a way that μ_{M_+} becomes a Poisson mapping.

Let u be the element defined by

$$(10.5) \quad u = \prod_{i=1}^{l'} \exp[t_i X_{-\gamma_i}] \in M_-, t_i = k_i \pmod{(q^{\frac{1}{2d}} - 1)},$$

where the product over roots is taken in the same order as in the normally ordered segment $\Delta_{\mathfrak{m}_+}$.

Denote by $p : \mathbb{C}_{\mathcal{A}'}[G^*] \rightarrow \mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*] = \mathbb{C}[G^*]$ the canonical projection. By Proposition 9.2 the elements $L^{\pm, V} = (p \otimes p_V)(q L^{\pm, V})$ belong to the space $\mathbb{C}[G^*] \otimes \text{End } \overline{V}$, where $p_V : V \rightarrow \overline{V} = V/(q^{\frac{1}{2d}} - 1)V$ is the projection of finite-dimensional $U_{\mathcal{A}}^s(\mathfrak{g})$ -module V onto the corresponding \mathfrak{g} -module \overline{V} , and the map

$$\mathbb{C}_{\mathcal{A}'}[G^*]/(q^{\frac{1}{2d}} - 1)\mathbb{C}_{\mathcal{A}'}[G^*] \rightarrow \mathbb{C}[G^*], \quad L^{\pm, V} \mapsto L^{\pm, \overline{V}}$$

is an isomorphism. In particular, from (4.10) it follows that

$$(10.6) \quad L^{-, \overline{V}} = (p \otimes id) \exp \left[\sum_{i=1}^l h H_i \otimes \pi_{\overline{V}}((- \frac{2s}{1-s} P_{\mathfrak{b}'} + P_{\mathfrak{b}'^\perp}) Y_i) \right] \times \prod_{\beta} \exp[p((1 - q_{\beta}^{-2}) e_{\beta}) \otimes \pi_{\overline{V}}(X_{-\beta})].$$

From (10.6) and the definition of χ^s we obtain that $\chi^s(\varphi) = \varphi(u)$ for every $\varphi \in \mathbb{C}[M_-]$. χ^s naturally extends to a character of the Poisson algebra $C^\infty(M_-)$.

Now applying Lemma 8.1 for $M = G^*$, $B = M_-$, $\pi = \mu_{M_+}$, $b = u$ we can define the reduced Poisson algebra $C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$ (see also Remark 8.4). Denote by I_u the ideal in $C^\infty(G^*)$ generated by elements $\mu_{M_+}^* \psi$, $\psi \in C^\infty(M_-)$, $\psi(u) = 0$. Let $P_u : C^\infty(G^*) \rightarrow C^\infty(G^*)/I_u = C^\infty(\mu_{M_+}^{-1}(u))$ be the canonical projection. Then the action (8.1) of $C^\infty(M_-)$ on $C^\infty(\mu_{M_+}^{-1}(u))$ takes the form:

$$(10.7) \quad \psi \cdot \varphi = P_u(\{\mu_{M_+}^* \psi, \tilde{\varphi}\}),$$

where $\psi \in C^\infty(M_-)$, $\varphi \in C^\infty(\mu_{M_+}^{-1}(u))$ and $\tilde{\varphi} \in C^\infty(G^*)$ is a representative of φ such that $P_u \tilde{\varphi} = \varphi$.

Lemma 10.1. $\mu_{M_+}^{-1}(u)$ is a subvariety in G^* . Moreover, the algebra $\mathbb{C}[G^*]/I$ is isomorphic to the algebra of regular functions on $\mu_{M_+}^{-1}(u)$, $\mathbb{C}[G^*]/I = \mathbb{C}[\mu_{M_+}^{-1}(u)]$, and the algebra $W^s(G)$ is isomorphic to the algebra of regular functions on the closure $\overline{q(\mu_{M_+}^{-1}(u))}$ of $q(\mu_{M_+}^{-1}(u))$ in G which are invariant with respect to the action (10.7) of $C^\infty(M_-)$ on $C^\infty(\mu_{M_+}^{-1}(u))$, i.e.

$$W^s(G) = \overline{\mathbb{C}[q(\mu_{M_+}^{-1}(u))]} \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}.$$

Proof. By definition $\mu_{M_+}^{-1}(u)$ is a subvariety in G^* . Next observe that $I = \mathbb{C}[G^*] \cap I_u$. Therefore by the definition of the algebra $\mathbb{C}[G^*]$ and of the map μ_{M_+} the quotient $\mathbb{C}[G^*]/I$ is identified with the algebra of regular functions on $\mu_{M_+}^{-1}(u)$.

Since $\mathbb{C}[M_-]$ is dense in $C^\infty(M_-)$ on every compact subset in M_- we have:

$$C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)} \cong C^\infty(\mu_{M_+}^{-1}(u))^{\mathbb{C}[M_-]}.$$

Finally observe that action (10.7) coincides with action (10.1) when restricted to regular functions, and that the image of the map $q : G^* \rightarrow G$ is open in G ; its closure coincides with G . Therefore by definition $Q_1 = \mathbb{C}[q(\mu_{M_+}^{-1}(u))]$ and $W^s(G) = (\mathbb{C}[G^*]/I)^{\mathbb{C}[M_-]} \cap Q_1$. \square

We shall realize the algebra $C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$ as the algebra of functions on a reduced Poisson manifold. In the spirit of Lemma 8.1 we shall construct a map that forms a dual pair together with

the mapping μ_{M_+} . In this construction we use the dressing action of the Poisson–Lie group G on G^* (see Proposition 8.2).

Consider the restriction of the dressing action $G^* \times G \rightarrow G^*$ to the subgroup $M_+ \subset G$. According to part (iv) of Proposition 7.2 $(\mathfrak{b}_+, \mathfrak{b}_-)$ is a subbialgebra of $(\mathfrak{g}, \mathfrak{g}^*)$. Therefore B_+ is a Poisson–Lie subgroup in G . We claim that $M_+ \subset B_+$ is an admissible subgroup.

Indeed, observe that since the roots $\gamma_1, \dots, \gamma_n$ are simple we have $\Delta_{s^1} = \{\gamma_1, \dots, \gamma_n\}$, and the complementary subset to $\Delta_{\mathfrak{m}_+}$ in Δ_+ is a minimal segment $\Delta_{\mathfrak{m}_+}^0$ with respect to normal ordering (5.8). Now using Proposition 7.2 (iv) the subspace \mathfrak{m}_+^\perp in \mathfrak{b}_- can be identified with the linear subspace in \mathfrak{b}_- spanned by the Cartan subalgebra \mathfrak{h} and the root subspaces corresponding to the roots from the minimal segment $-\Delta_{\mathfrak{m}_+}^0$. Using the fact that the adjoint action of \mathfrak{h} normalizes root subspaces and Lemma 5.2 we deduce that $\mathfrak{m}_+^\perp \subset \mathfrak{b}_-$ is a Lie subalgebra, and hence $M_+ \subset B_+$ is an admissible subgroup. Therefore $C^\infty(G^*)^{M_+}$ is a Poisson subalgebra in the Poisson algebra $C^\infty(G^*)$.

Proposition 10.2. ([32], **Proposition 11.2**) *Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple. Then the algebra $C^\infty(G^*)^{M_+}$ is the centralizer of $\mu_{M_+}^*(C^\infty(M_-))$ in the Poisson algebra $C^\infty(G^*)$.*

Proof. First recall that, as we observed above, B_+ is a Poisson–Lie subgroup in G . By Proposition 8.2 for $X \in \mathfrak{b}_+$ we have:

$$(10.8) \quad L_{\widehat{X}}\varphi(L_+, L_-) = (\theta_{G^*}(L_+, L_-), X)(\xi_\varphi) = (r_-^{-1}\mu_{B_+}^*(\theta_{B_-}), X)(\xi_\varphi),$$

where \widehat{X} is the corresponding vector field on G^* , ξ_φ is the Hamiltonian vector field of $\varphi \in C^\infty(G^*)$, and the map $\mu_{B_+} : G^* \rightarrow B_-$ is defined by $\mu_{B_+}(L_+, L_-) = L_-$. Now from Proposition 7.2 (iv) and the definition of the moment map it follows that μ_{B_+} is a moment map for the dressing action of the subgroup B_+ on G^* .

We also proved above that M_+ is an admissible subgroup in the Lie–Poisson group B_+ . Moreover the dual group B_- can be uniquely factorized as $B_- = M_+^\perp M_-$, where $M_+^\perp \subset B_-$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{m}_+^\perp \subset \mathfrak{b}_-$.

We conclude that all the conditions of Proposition 8.4 are satisfied with $A = B_+$, $K = M_+$, $A^* = B_-$, $T = M_-$, $K^\perp = M_+^\perp$, $\mu = \mu_{B_+}$. It follows that the algebra $C^\infty(G^*)^{M_+}$ is the centralizer of $\mu_{M_+}^*(C^\infty(M_-))$ in the Poisson algebra $C^\infty(G^*)$. This completes the proof. \square

Let G^*/M_+ be the quotient of G^* with respect to the dressing action of M_+ , $\pi : G^* \rightarrow G^*/M_+$ the canonical projection. Note that the space G^*/M_+ is not a smooth manifold. However, in the next section we will see that the subspace $\pi(\mu_{M_+}^{-1}(u)) \subset G^*/M_+$ is a smooth manifold. Therefore by Remark 8.5 the algebra $C^\infty(\pi(\mu_{M_+}^{-1}(u)))$ is isomorphic to $C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$.

11. CROSS-SECTION THEOREM

In this section, following [32], we describe the reduced space $\pi(\mu_{M_+}^{-1}(u)) \subset G^*/M_+$ and the algebra $W^s(G)$ assuming again that the roots $\gamma_1, \dots, \gamma_n$ are simple or that the set $\gamma_1, \dots, \gamma_n$ is empty.

First observe that using the map $q : G^* \rightarrow G$ and Proposition 8.2 one can reduce the study of the dressing action to the study of the action of G on itself by conjugations. This simplifies many geometric problems. Consider the restriction of this action to the subgroup M_+ . Denote by $\pi_q : G \rightarrow G/M_+$ the canonical projection onto the quotient with respect to this action.

First we describe the image of the “level surface” $\mu_{M_+}^{-1}(u)$ under the map q . Let $X_\alpha(t) = \exp(tX_\alpha) \in G$, $t \in \mathbb{C}$ be the one-parametric subgroup in the algebraic group G corresponding to root $\alpha \in \Delta$. Recall that for any $\alpha \in \Delta_+$ and any $t \neq 0$ the element $s_\alpha(t) = X_\alpha(-t)X_{-\alpha}(t)X_\alpha(-t) \in G$ is a representative for the reflection s_α corresponding to the root α . Denote by $s \in G$ the following representative of the Weyl group element $s \in W$,

$$(11.1) \quad s = s_{\gamma_1}(t_1) \dots s_{\gamma_{l'}}(t_{l'}),$$

where the numbers t_i are defined in (10.5), and we assume that $t_i \neq 0$ for any i .

We shall also use the following representatives for s^1 and s^2

$$s^1 = s_{\gamma_1}(t_1) \dots s_{\gamma_n}(t_n), \quad s^2 = s_{\gamma_{n+1}}(t_{n+1}) \dots s_{\gamma_{l'}}(t_{l'}).$$

Let Z be the subgroup of G corresponding to the Lie subalgebra \mathfrak{z} generated by the semisimple part \mathfrak{m} of the Levi subalgebra \mathfrak{l} and by the centralizer of s in \mathfrak{h} . Denote by N the subgroup of G corresponding to the Lie subalgebra \mathfrak{n} and by \overline{N} the opposite unipotent subgroup in G with the Lie algebra $\overline{\mathfrak{n}} = \bigoplus_{m < 0} (\mathfrak{g})_m$. By definition we have that $N_+ \subset ZN$.

Proposition 11.1. ([32], **Proposition 12.1**) *Let $q : G^* \rightarrow G$ be the map introduced in Proposition 8.2,*

$$q(L_+, L_-) = L_- L_+^{-1}.$$

Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Then $q(\mu_{M_+}^{-1}(u))$ is a subvariety in $NsZN$ and the closure $\overline{q(\mu_{M_+}^{-1}(u))}$ of $q(\mu_{M_+}^{-1}(u))$ is also contained in $NsZN$.

We identify $q(\mu_{M_+}^{-1}(u))$ with the subvariety in $NsZN$ described in the previous proposition.

Proposition 11.2. ([31], **Propositions 2.1 and 2.2**) *Let $N_s = \{v \in N \mid sv s^{-1} \in \overline{N}\}$. Then the conjugation map*

$$(11.2) \quad N \times sZN_s \rightarrow NsZN$$

is an isomorphism of varieties. Moreover, the variety sZN_s is a transversal slice to the set of conjugacy classes in G .

Theorem 11.3. *Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or the set $\gamma_1, \dots, \gamma_n$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Then the (locally defined) conjugation action of M_+ on $q(\mu_{M_+}^{-1}(u))$ is (locally) free, the quotient $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is a smooth variety and the algebra of regular functions on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is isomorphic to the algebra of regular functions on the slice sZN_s .*

The Poisson algebra $W^s(G)$ is isomorphic to the Poisson algebra of regular functions on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$, $W^s(G) = \mathbb{C}[\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})] = \mathbb{C}[sZN_s]$; the algebra $\mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}]$ is isomorphic to $\mathbb{C}[M_+] \otimes W^s(G) \cong \mathbb{C}[M_+] \otimes \mathbb{C}[sZN_s]$. Thus the algebra $W_q^s(G)$ is a noncommutative deformation of the algebra of regular functions on the transversal slice sZN_s .

Proof. First observe that by construction $q(\mu_{M_+}^{-1}(u)) \subset NsZN$ is (locally) stable under the action of $M_+ \subset N$ on $NsZN$ by conjugations. Since the conjugation action of N on $NsZN$ is free the (locally defined) conjugation action of M_+ on $q(\mu_{M_+}^{-1}(u))$ is (locally) free as well. Therefore the quotient $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is a smooth variety.

Since by Proposition 11.1 $\overline{q(\mu_{M_+}^{-1}(u))} \subset NsZN$ the induced (local) action of M_+ on $\overline{q(\mu_{M_+}^{-1}(u))}$ is (locally) free as well. Now observe that from the description of the set $\overline{q(\mu_{M_+}^{-1}(u))}$ given in the end of Proposition 11.1 it follows that $sZN_s \subset \overline{q(\mu_{M_+}^{-1}(u))}$. Since by the previous proposition any two points of sZN_s are not N -conjugate they are not M_+ -conjugate as well, and hence we have an embedding $sZN_s \subset \pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$. From formula (5.10) for the cardinality $\sharp \Delta_{\mathfrak{m}_+}$ of the set $\Delta_{\mathfrak{m}_+}$ and from the definition of $q(\mu_{M_+}^{-1}(u))$ we deduce that the dimension of the quotient $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$

is equal to the dimension of the variety sZN_s ,

$$\begin{aligned} \dim \pi_q(\overline{q(\mu_{M_+}^{-1}(u))}) &= \dim G - 2\dim M_+ = 2D + l - 2\sharp\Delta_{\mathfrak{m}_+} = 2D + l - \\ &- 2(D - \frac{l(s) - l'}{2} - D_0) = l(s) + 2D_0 + l - l' = \dim N_s + \dim Z = \dim sZN_s. \end{aligned}$$

Note also that both sZN_s and $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ are connected. Therefore any regular function on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is completely defined by its restriction to sZN_s . Such restrictions also span the space of regular functions on sZN_s since by the previous proposition the restriction map induces an isomorphism of the space of N -invariant regular functions on $NsZN$ and the space of regular functions on sZN_s , and N -invariant regular functions on $NsZN$ can be restricted to M_+ -invariant functions on $\overline{q(\mu_{M_+}^{-1}(u))}$. Therefore the algebra of regular functions on $\pi_q(\overline{q(\mu_{M_+}^{-1}(u))})$ is isomorphic to the algebra of regular functions on the slice sZN_s . This proves the first statement of the proposition.

Now observe that by Remark 8.5 the map

$$C^\infty(\pi(\mu_{M_+}^{-1}(u))) \rightarrow C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}, \quad \psi \mapsto \pi^*\psi$$

is an isomorphism. By construction the map $\pi_q : \overline{q(\mu_{M_+}^{-1}(u))} \rightarrow \pi_q \overline{q(\mu_{M_+}^{-1}(u))}$ is a morphism of varieties. Therefore the map

$$\mathbb{C}[\pi_q \overline{q(\mu_{M_+}^{-1}(u))}] \rightarrow \mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}] \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}, \quad \psi \mapsto \pi_q^*\psi$$

is an isomorphism.

Finally observe that by Lemma 10.1 the algebra $\mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}] \cap C^\infty(\mu_{M_+}^{-1}(u))^{C^\infty(M_-)}$ is isomorphic to $W^s(G)$, and hence $W^s(G) \cong \mathbb{C}[\pi_q \overline{q(\mu_{M_+}^{-1}(u))}]$.

The first three statements of the theorem also imply isomorphisms, $\mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}] \cong \mathbb{C}[M_+] \otimes W^s(G) \cong \mathbb{C}[M_+] \otimes \mathbb{C}[sZN_s]$. This completes the proof. \square

Remark 11.8. *A similar theorem can be proved in case when the roots $\gamma_{n+1}, \dots, \gamma_\nu$ are simple or the set $\gamma_{n+1}, \dots, \gamma_\nu$ is empty. In that case instead of the map $q : G^* \rightarrow G$ one should use another map $q' : G^* \rightarrow G$, $q'(L_+, L_-) = L_-^{-1}L_+$ which has the same properties as q , see [26], Section 2.*

Theorem 11.3 implies that the algebra $W^s(G)$ coincides with the deformed Poisson W -algebra introduced in [31].

In conclusion we discuss a simple property of the algebra $W_\varepsilon^s(G)$ which allows to construct non-commutative deformations of coordinate rings of singularities arising in the fibers of the conjugation quotient map $\delta_G : G \rightarrow H/W$ generated by the inclusion $\mathbb{C}[H]^W \simeq \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$, where H is the maximal torus of G corresponding to the Cartan subalgebra \mathfrak{h} and W is the Weyl group of the pair (G, H) .

Observe that each central element $z \in Z(\mathbb{C}_\varepsilon[G_*])$ obviously gives rise to an element $\rho_{\chi_\varepsilon^s}(z) \in \mathbb{C}_\varepsilon[G_*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$, and since z is central

$$\begin{aligned} \rho_{\chi_\varepsilon^s}(z) &\in \text{Hom}_{\mathbb{C}_\varepsilon[M_-]}(\mathbb{C}_{\chi_\varepsilon^s}, \mathbb{C}_\varepsilon[G_*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s})^{opp} = \\ &= \text{End}_{\mathbb{C}_\varepsilon[G_*]}(\mathbb{C}_\varepsilon[G_*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s})^{opp} = W_\varepsilon^s(G). \end{aligned}$$

The proof of the following proposition is similar to that of Theorem A_h in [29].

Proposition 11.4. *Let $\varepsilon \in \mathbb{C}$ be generic. Then the restriction of the linear map $\rho_{\chi_\varepsilon^s} : \mathbb{C}_\varepsilon[G_*] \rightarrow \mathbb{C}_\varepsilon[G_*] \otimes_{\mathbb{C}_\varepsilon[M_-]} \mathbb{C}_{\chi_\varepsilon^s}$ to the center $Z(\mathbb{C}_\varepsilon[G_*])$ of $\mathbb{C}_\varepsilon[G_*]$ gives rise to an injective homomorphism of algebras,*

$$\rho_{\chi_\varepsilon^s} : Z(\mathbb{C}_\varepsilon[G_*]) \rightarrow W_\varepsilon^s(G).$$

Now if $\eta : Z(\mathbb{C}_\varepsilon[G_*]) \rightarrow \mathbb{C}$ is a character then from Theorem 11.3 and the results of Section 6 in [31] it follows that the algebra $W_\varepsilon^s(G)/W_\varepsilon^s(G)\ker \eta$ can be regarded as a noncommutative deformation of the algebra of regular functions defined on a fiber of the conjugation quotient map $\delta_G : sZN_s \rightarrow H/W$. In particular, for singular fibers we obtain noncommutative deformations of the coordinate rings of the corresponding singularities.

12. SKRYABIN EQUIVALENCE FOR EQUIVARIANT MODULES OVER QUANTUM GROUPS

In this section we establish a remarkable equivalence between the category of $W_\varepsilon^s(G)$ -modules and a certain category of $\mathbb{C}_\varepsilon[G_*]$ modules. This equivalence is a quantum group counterpart of Skryabin equivalence established in the Appendix to [21].

Let $J = \text{Ker } \varepsilon|_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}$ be the augmentation ideal of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ related to the counit ε of $U_{\mathcal{A}'}^s(\mathfrak{g})$, and $\mathbb{C}_{\mathcal{A}'}$ the trivial representation of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ given by the counit. Let V be a finitely generated $\mathbb{C}_{\mathcal{A}'}[G_*]$ -module which satisfies the following conditions:

- (1) V is free as an \mathcal{A}' -module.
- (2) V is a right $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module with respect to an action Ad such that the action of the augmentation ideal J on V is locally nilpotent.
- (3) The following compatibility condition holds for the two actions

$$(12.1) \quad \text{Ad}x(yv) = \text{Ad}x_2(y)\text{Ad}x_1(v), \quad x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+), \quad y \in \mathbb{C}_{\mathcal{A}'}[G_*], \quad v \in V,$$

where $\Delta_s(x) = x_1 \otimes x_2$, $\text{Ad}x(y)$ is the adjoint action of $x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $y \in \mathbb{C}_{\mathcal{A}'}[G_*]$.

An element $v \in V$ is called a Whittaker vector if $\text{Ad}xv = \varepsilon(x)v$ for any $x \in U_{\mathcal{A}'}^s(\mathfrak{m}_+)$. The space

$$(12.2) \quad \text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathbb{C}_{\mathcal{A}'}, V) = \text{Wh}(V).$$

is called the space of Whittaker vectors of V .

Consider the induced $U_{\mathcal{A}'}^s(\mathfrak{g})$ -module $W = U_{\mathcal{A}'}^s(\mathfrak{g}) \otimes_{\mathbb{C}_{\mathcal{A}'}[G_*]} V$. Using the adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{g})$ on itself one can naturally extend the adjoint action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ from V to W in such a way that compatibility condition (12.1) is satisfied for the natural action of $U_{\mathcal{A}'}^s(\mathfrak{g})$ and the adjoint action Ad of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on W . As we observed in Section 9 (see formula (9.15)) $\Delta_s^{opp}(U_{\mathcal{A}'}^s(\mathfrak{m}_+)) \subset U_{\mathcal{A}'}^s(\mathfrak{b}_+) \otimes U_{\mathcal{A}'}^s(\mathfrak{m}_+)$. From this using the fact that the elements $\tilde{e}_\beta = (1 - q_\beta^{-2})e_\beta$, $e_\beta \in U_{\mathcal{A}'}^s(\mathfrak{m}_+)$, are generators of $\mathbb{C}_{\mathcal{A}'}[M_-]$ one immediately deduces that $\Delta_s^{opp}(\mathbb{C}_{\mathcal{A}'}[M_-]) \subset U_{\mathcal{A}'}^s(\mathfrak{b}_+) \otimes \mathbb{C}_{\mathcal{A}'}[M_-]$. In fact $\Delta_s^{opp}(\mathbb{C}_{\mathcal{A}'}[M_-]) \subset \mathbb{C}_{\mathcal{A}'}[B_-] \otimes \mathbb{C}_{\mathcal{A}'}[M_-]$ since $\mathbb{C}_{\mathcal{A}'}[M_-] \subset \mathbb{C}_{\mathcal{A}'}[B_-]$ which is a Hopf algebra.

We shall require that

- (4) For any $x \in \mathbb{C}_{\mathcal{A}'}[M_-]$ the natural action of the element $(S^{-1} \otimes \chi_q^s)\Delta^{opp}(x) \in \mathbb{C}_{\mathcal{A}'}[G_*]$ on W coincides with the adjoint action $\text{Ad}x$ of x on W .

As in the second part of the proof of Proposition 9.4 one can see that the last condition implies that for any $z \in \mathbb{C}_{\mathcal{A}'}[G_*] \cap I_q$ and $v \in \text{Wh}(V)$ $zv = 0$.

Denote by $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_loc}^{\chi_q^s}$ the category of finitely generated $\mathbb{C}_{\mathcal{A}'}[G_*]$ -module which satisfy conditions 1–4. Morphisms in the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_loc}^{\chi_q^s}$ are $\mathbb{C}_{\mathcal{A}'}[G_*]$ - and $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module homomorphisms. We call $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_loc}^{\chi_q^s}$ the category of $(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \chi_q^s)$ -equivariant modules over $\mathbb{C}_{\mathcal{A}'}[G_*]$.

Note that the algebra $W_q^s(G)$ naturally acts in the space of Whittaker vectors for any object V of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_loc}^{\chi_q^s}$. Indeed, if $w, w' \in \mathbb{C}_{\mathcal{A}'}[G_*]$ are two representatives of an

element from $W_q^s(G)$ then $w - w' \in \mathbb{C}_{\mathcal{A}'}[G_*] \cap I_q$, and hence for any $v \in \text{Wh}(V)$ $wv = w'v$. Moreover, by the definition of the algebra $W_q^s(G)$ and by condition (12.1) we have

$$\text{Ad}x(wv) = \text{Ad}x_2(w)\text{Ad}x_1(v) = \text{Ad}x_2(w)\varepsilon(x_1)v = \text{Ad}x(w)v = \varepsilon(x)wv.$$

Therefore wv is a Whittaker vector independent of the choice of the representative w .

Proposition 12.1. *For any finitely generated $W_q^s(G)$ -module E which is free as an \mathcal{A}' -module the space $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E$ is an object in $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_\text{loc}}^{\chi_q^s}$, and*

$$\text{Wh}(Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E) = \text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathbb{C}_{\mathcal{A}'}, Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E) = E.$$

Proof. First we prove that $Q_{\mathcal{A}'}$ is an object in $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_\text{loc}}^{\chi_q^s}$. We shall prove that the adjoint action of the augmentation ideal J of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'}$ is locally nilpotent. All the other properties of objects of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_\text{loc}}^{\chi_q^s}$ for $Q_{\mathcal{A}'}$ were already established in Proposition 9.5.

Indeed, let $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ be the subspace in $\text{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}(\mathfrak{m}_+), W_q^s(G))$ which consists of the linear maps vanishing on some power of the augmentation ideal $J = \text{Ker } \varepsilon$ of $U_{\mathcal{A}'}(\mathfrak{m}_+)$, $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}(\mathfrak{m}_+), W_q^s(G)) = \{f \in \text{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}(\mathfrak{m}_+), W_q^s(G)) : f(J^n) = 0 \text{ for some } n > 0\}$. Fix any linear map $\rho : Q_{\mathcal{A}'} \rightarrow W_q^s(G) \subset Q_{\mathcal{A}'}$ the restriction of which to $W_q^s(G)$ is the identity map, and let for any $v \in Q_{\mathcal{A}'}$ $\sigma(v) : U_{\mathcal{A}'}^s(\mathfrak{m}_+) \rightarrow W_q^s(G)$ be the \mathcal{A}' -linear homomorphism given by $\sigma(v)(x) = \rho(\text{Ad}x(v))$. Since the adjoint action of $U_{\mathcal{A}'}(\mathfrak{m}_+)$ on $\mathbb{C}_{\mathcal{A}'}[G_*]$ is locally finite the induced adjoint action of $U_{\mathcal{A}'}(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'}$ is locally finite as well (see the arguments in the end of the proof of Proposition 9.3). Therefore for any $v \in Q_{\mathcal{A}'}$ the space $\text{Ad}U_{\mathcal{A}'}(\mathfrak{m}_+)(v)$ has finite rank over \mathcal{A}' . This implies that in fact $\sigma(v) \in \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$, and we have a map $\sigma : Q_{\mathcal{A}'} \rightarrow \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$.

By definition σ is a homomorphism of right $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -modules, where the right action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ is induced by multiplication in $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ from the left.

We claim that σ is injective. Indeed, consider the specialization σ_1 of the homomorphism σ at $q = 1$. The specialization of the algebra $U_{\mathcal{A}'}(\mathfrak{m}_+)$ at $q = 1$ is isomorphic to $U(\mathfrak{m}_+)$, and the specialization $Q_1 = Q_{\mathcal{A}'} / (q^{\frac{1}{2d}} - 1)Q_{\mathcal{A}'}$ of the $U_{\mathcal{A}'}(\mathfrak{m}_+)$ -module $Q_{\mathcal{A}'}$ at $q = 1$ is isomorphic to $\mathbb{C}[q(\mu_{M_+}^{-1}(u))]$. By Theorem 11.3 $\mathbb{C}[q(\mu_{M_+}^{-1}(u))] \cong \mathbb{C}[M_+] \otimes W^s(G)$. From Proposition 10.2 we obtain that the induced action of $U(\mathfrak{m}_+)$ on the corresponding variety $q(\mu_{M_+}^{-1}(u))$ is induced by the conjugation action of M_+ and now using Proposition 11.2 one immediately deduces that the induced action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+] \otimes W^s(G)$ is generated by the action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+]$ by left invariant differential operators.

Using the exponential map $\exp : \mathfrak{m}_+ \rightarrow M_+$ we can also identify $\mathbb{C}[M_+] \otimes W^s(G)$ with the right $U(\mathfrak{m}_+)$ -module $\text{hom}_{\mathbb{C}}(U(\mathfrak{m}_+), W^s(G)) = \{f \in \text{Hom}_{\mathbb{C}}(U(\mathfrak{m}_+), W^s(G)) : f(J_1^n) = 0 \text{ for some } n > 0\}$, where J_1 is the augmentation ideal of $U(\mathfrak{m}_+)$ generated by \mathfrak{m}_+ , and the right action of $U(\mathfrak{m}_+)$ on $\text{hom}_{\mathbb{C}}(U(\mathfrak{m}_+), W^s(G))$ is induced by multiplication in $U(\mathfrak{m}_+)$ from the left.

On the other hand the specialization of $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ at $q = 1$ is also isomorphic to $\text{hom}_{\mathbb{C}}(U(\mathfrak{m}_+), W^s(G))$, and hence under the above identifications the specialization of σ_1 of map σ at $q = 1$ becomes the identity map.

Now let W be the kernel of σ , and $W_1 \subset Q_1$ its image under the canonical projection $Q_{\mathcal{A}'} \rightarrow Q_1 = Q_{\mathcal{A}'} / (q^{\frac{1}{2d}} - 1)Q_{\mathcal{A}'}$. W_1 must be contained in the kernel of σ_1 . Since this kernel is trivial W_1 must be trivial as well, and hence $W = (q^{\frac{1}{2d}} - 1)W'$, $W' \subset Q_{\mathcal{A}'}$. Since $Q_{\mathcal{A}'}$ is \mathcal{A}' -free and \mathcal{A}' has no zero divisors we also have $W' \subset W$. Iterating this process we deduce that any element $w \in W$ can be represented in the form $w = (q^{\frac{1}{2d}} - 1)^B w'$, $w' \in W$ with arbitrary large $B \in \mathbb{N}$ which is possible only in case when $W = 0$. Therefore σ is injective.

Thus $Q_{\mathcal{A}'}$ is a submodule of $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ the action of J on which is locally nilpotent. Therefore the action of J on $Q_{\mathcal{A}'}$ is locally nilpotent as well.

We conclude that for any finitely generated $W_q^s(G)$ -module E which is free as an \mathcal{A}' -module the space $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E$ can be equipped with the adjoint action induced by the adjoint action on $Q_{\mathcal{A}'}$ in such a way that the compatibility condition (12.1) is satisfied. Since the adjoint action of the augmentation ideal J on $Q_{\mathcal{A}'}$ is locally nilpotent the induced adjoint action of the augmentation ideal J on $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E$ is locally nilpotent as well.

The fact that $Q_{\mathcal{A}'}$ is an object of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$ implies now that $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E$ is an object of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$ as well. Moreover, by the definition of the algebra $W_q^s(G)$

$$(12.3) \quad \text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathbb{C}_{\mathcal{A}'}, Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E) = \text{Wh}(Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E) = W_q^s(G) \otimes_{W_q^s(G)} E = E.$$

This completes the proof of the fact that $Q_{\mathcal{A}'}$ and $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} E$ are objects of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$. □

Obviously we also have that for any object V of the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$ the canonical map $Q_{\mathcal{A}'} \otimes_{W_q^s(G)} \text{Wh}(V) \rightarrow V$ is a morphism in the category $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$.

We also denote by $\mathbb{C}_\varepsilon[G_*] - \text{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{\chi_\varepsilon^s}$ the category of $\mathbb{C}_\varepsilon[G_*]$ -modules which are specializations of modules from $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$ at $q = \varepsilon \in \mathbb{C}$. The spaces of Whittaker vectors for modules from $\mathbb{C}_\varepsilon[G_*] - \text{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{\chi_\varepsilon^s}$, the adjoint action and the canonical map $Q_\varepsilon \otimes_{W_\varepsilon^s(G)} \text{Wh}(V) \rightarrow V$, $V \in \mathbb{C}_\varepsilon[G_*] - \text{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{\chi_\varepsilon^s}$ are defined similarly to the case of modules from $\mathbb{C}_{\mathcal{A}'}[G_*] - \text{mod}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)_{loc}}^{\chi_q^s}$.

We have the following obvious ε -specialization of Proposition 12.1.

Proposition 12.2. *Let $\varepsilon \in \mathbb{C}$ be generic. Then for any finitely generated $W_\varepsilon^s(G)$ -module E the space $Q_\varepsilon \otimes_{W_\varepsilon^s(G)} E$ is an object in $\mathbb{C}_\varepsilon[G_*] - \text{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{\chi_\varepsilon^s}$, and*

$$\text{Wh}(Q_\varepsilon \otimes_{W_\varepsilon^s(G)} E) = \text{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, Q_\varepsilon \otimes_{W_\varepsilon^s(G)} E) = E,$$

where \mathbb{C}_ε is the trivial representation of $U_\varepsilon^s(\mathfrak{m}_+)$ given by the counit.

The following proposition is crucial for the proof of the main statement of this paper.

Proposition 12.3. *Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Then for generic $\varepsilon \in \mathbb{C}$ Q_ε is isomorphic to $\text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes W_\varepsilon^s(G)$ as a $U_\varepsilon^s(\mathfrak{m}_+) - W_\varepsilon^s(G)$ -bimodule, where $\text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})$ is the subspace in $\text{Hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})$ which consists of the linear maps vanishing on some power of the augmentation ideal $J = \text{Ker } \varepsilon$ (here ε is the counit of $U_\varepsilon^s(\mathfrak{g})$) of $U_\varepsilon^s(\mathfrak{m}_+)$, $\text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) = \{f \in \text{Hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) : f(J^n) = 0 \text{ for some } n > 0\}$.*

Proof. First we show that the specialization $\sigma_\varepsilon : Q_\varepsilon \rightarrow \text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ at $q = \varepsilon$ of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module homomorphism $\sigma : Q_{\mathcal{A}'} \rightarrow \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ constructed in the proof of Proposition 12.1 is an isomorphism of right $U_\varepsilon^s(\mathfrak{m}_+)$ -modules.

First we prove that σ_ε is injective. The proof will be based on the following lemma that will be also used later.

Lemma 12.4. *Let $\phi : X \rightarrow Y$ be a homomorphism of $U_\varepsilon^s(\mathfrak{m}_+)$ -modules. Assume that the action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_+)$ on X is locally nilpotent and that the restriction of ϕ to the subspace of Whittaker vectors of X is injective. Then ϕ is injective.*

Proof. Let $Z \subset X$ be the kernel of ϕ . Assume that Z is not trivial. Observe that Z is invariant with respect to the action induced by the action of $U_\varepsilon^s(\mathfrak{m}_+)$ on X , and that the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_+)$ acts on X by locally nilpotent transformations. Therefore by Engel theorem Z must contain a nonzero $U_\varepsilon^s(\mathfrak{m}_+)$ -invariant vector which is a Whittaker vector $v \in X$. But since the restriction of ϕ to the subspace of Whittaker vectors of X is injective $\phi(v) \neq 0$. Thus we arrive at a contradiction, and hence ϕ is injective. \square

Now we prove that σ_ε is injective. Observe that by Proposition 12.2 the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_+)$ acts on Q_ε by locally nilpotent transformations. Let $v \in W_\varepsilon^s(G)$ be a nonzero Whittaker vector of Q_ε . By the definition of map σ_ε we have $\sigma_\varepsilon(v)(1) = \rho_\varepsilon(v) = v$, where $\rho_\varepsilon : Q_\varepsilon \rightarrow W_\varepsilon^s(G) \subset Q_\varepsilon$ is the linear map used in the definition of the map σ_ε the restriction of which to $W_q^s(G)$ is the identity map. Therefore $\sigma_\varepsilon(v) \neq 0$. Now by Lemma 12.4 applied to $\sigma_\varepsilon : Q_\varepsilon \rightarrow \text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ the homomorphism σ_ε is injective.

Now we prove that σ_ε is surjective. In order to do that we shall calculate the cohomology space of the right $U_\varepsilon^s(\mathfrak{m}_+)$ -module Q_ε with respect to the adjoint action of $U_\varepsilon^s(\mathfrak{m}_+)$,

$$(12.4) \quad \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_\varepsilon, Q_\varepsilon).$$

We shall show that

$$(12.5) \quad \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^n(\mathbb{C}_\varepsilon, Q_\varepsilon) = 0, \quad n > 0.$$

Note that we already know that by definition

$$(12.6) \quad \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^0(\mathbb{C}_\varepsilon, Q_\varepsilon) = \text{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, Q_\varepsilon) = W_\varepsilon^s(G).$$

We shall calculate the Ext functors in formula (12.5) using a deformation argument which is based on upper semicontinuity of cohomology functor with respect to base ring localizations discovered by Grothendieck (see for instance [38], Theorem 1.2 for the formulation of this principle suitable for our purposes). Let X^\bullet be a complex of finitely generated free modules over a ring \mathbf{k} , X_p^\bullet the corresponding complex over the residue field $k(p)$ of the localization of \mathbf{k} at a prime ideal p . Then for each i the function $p \mapsto \dim_{\mathbb{C}} H^i(X_p^\bullet)$ is upper semicontinuous on $\text{Spec}(\mathbf{k})$. In particular, if $H^i(X_{p_0}^\bullet) = 0$ for some p_0 then for generic p we have $H^i(X_p^\bullet) = 0$.

As \mathbf{k} we shall take \mathcal{A}' . Note that one can define a localization, $\mathcal{A}'/(1 - q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$ as well as similar localizations for other generic values of ε , $\mathcal{A}'/(\varepsilon^{\frac{1}{2d}} - q^{\frac{1}{2d}})\mathcal{A}' = \mathbb{C}$.

An appropriate complex X^\bullet is a little bit more complicated to define. Let $\mathbb{C}_{\mathcal{A}'}$ be the trivial representation of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ given by the counit. We shall construct a complex $X_{\mathcal{A}'}^\bullet$ for calculating the functor $\text{Ext}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_{\mathcal{A}'}, Q_{\mathcal{A}'})$ the specialization of which for any generic ε is a complex for calculating the functor $\text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_\varepsilon, Q_\varepsilon)$, and the specialization of $X_{\mathcal{A}'}^\bullet$ at $q = 1$ is a complex for calculating $U(\mathfrak{m}_+)$ -cohomology with values $\mathbb{C}[M_+] \otimes W^s(G)$, where the action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+] \otimes W^s(G)$ is induced by the natural action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+]$ by left invariant differential operators. These cohomology is just the de Rham cohomology of M_+ , and hence is trivial in nonzero degrees. Moreover, the complex $X_{\mathcal{A}'}^\bullet$ will be filtered by finitely generated free modules. Therefore Grothendieck upper semicontinuity of cohomology together with the property of the specialization of our complex at $q = 1$ imply vanishing property (12.5).

To construct the complex $X_{\mathcal{A}'}^\bullet$ we first recall the definition of the standard bar resolution of an associative algebra A over a ring \mathbf{k} regarded as an $A - A$ -bimodule (see [39], Ch. 9, §6),

$$(12.7) \quad \begin{aligned} \text{Bar}^n(A) &= \underbrace{A \otimes_{\mathbf{k}} \dots \otimes_{\mathbf{k}} A}_{n+2 \text{ times}}, \quad n \geq 0, \\ d(a_0 \otimes \dots \otimes a_{n+1}) &= \\ \sum_{s=0}^n (-1)^s a_0 \otimes \dots \otimes a_s a_{s+1} \otimes \dots \otimes a_{n+1} \end{aligned}$$

where $a_0, \dots, a_{n+1} \in A$.

Now observe that if one introduces degrees of elements of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ by putting $\text{dege}_i = 1$, $i = 1, \dots, l$ the algebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ becomes naturally \mathbb{N} -graded by subspaces $U_{\mathcal{A}'}^s(\mathfrak{m}_+)^k$ which are free over \mathcal{A}' and have finite rank over \mathcal{A}' . Let $U_{\mathcal{A}'}^s(\mathfrak{m}_+)^k$ be the induced grading of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ and denote by $U_{\mathcal{A}'}^s(\mathfrak{m}_+)_k$ the induced filtration of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ by subspaces of finite rank over \mathcal{A}' .

Now one can define a filtration of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $Q_{\mathcal{A}'}$ by free \mathcal{A}' -modules of finite rank over \mathcal{A}' . In order to do that we recall that $Q_{\mathcal{A}'}$ is a submodule of $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ as we observed in the proof of Proposition 12.1. We also observe that from the definition of the space $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))$ it follows that

$$(12.8) \quad \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G)) = \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}') \otimes_{\mathcal{A}'} W_q^s(G),$$

where

$$\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}') = \oplus_{k \leq 0} \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+)^{-k}, \mathcal{A}'),$$

Observe that $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')$ is naturally a \mathbb{Z} -graded module over the \mathbb{N} -graded algebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$. Denote by $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')_k = \oplus_{p \geq k} \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+)^{-p}, \mathcal{A}')$ the corresponding filtration of $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')$ by subspaces which are free over \mathcal{A}' and have finite rank over \mathcal{A}' . By construction the action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')$ preserves the filtration of $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')$. Combining the filtration on $\text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')$ with an arbitrary filtration $W_q^s(G)_k$, $k \in \mathbb{N}$ of $W_q^s(G)$ by free \mathcal{A}' -submodules of finite rank we obtain a filtration of

$$\begin{aligned} \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G)) &= \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}') \otimes_{\mathcal{A}'} W_q^s(G), \\ \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W_q^s(G))_k &= \bigcup_{q-p \leq k} \text{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), \mathcal{A}')_p \otimes_{\mathcal{A}'} W_q^s(G)_q, \quad k \in \mathbb{N}. \end{aligned}$$

The induced filtration of the submodule $Q_{\mathcal{A}'}$ has components which are free \mathcal{A}' -modules of finite rank. By construction the action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $Q_{\mathcal{A}'}$ preserves the components $(Q_{\mathcal{A}'})_k$ of that filtration.

The filtration $U_{\mathcal{A}'}^s(\mathfrak{m}_+)_k$ induces a filtration $\text{Bar}^n(U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k$ of the complex $\text{Bar}^n(U_{\mathcal{A}'}^s(\mathfrak{m}_+))$ by subcomplexes with finite rank graded components.

Consider the subcomplex

$$\begin{aligned} &\text{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'}) = \\ &= \bigcup_k \text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k, Q_{\mathcal{A}'}) \end{aligned}$$

of the complex $\text{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$. Since $\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+))$ is homotopic to $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ as a filtered $U_{\mathcal{A}'}^s(\mathfrak{m}_+) - U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ bimodule the cohomology of

$$\text{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$$

coincide with $Q_{\mathcal{A}'}$. We claim that the homological degree graded components of

$$\text{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$$

are injective $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -modules, and hence $\mathrm{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$ is an injective resolution of $Q_{\mathcal{A}'}$.

Indeed, by construction each of the components $\mathrm{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathrm{Bar}^n(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$ is isomorphic to $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) = \bigcup_k \mathrm{Hom}_{\mathcal{A}'}((U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k, W)$ for some free \mathcal{A}' -module W , and the right action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ on $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ is induced by multiplication on $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ from the left. Clearly, $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ is the subspace of $\mathrm{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ which consist of the linear maps vanishing on some power of the augmentation ideal $J = \mathrm{Ker} \varepsilon$ of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$, $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) = \{f \in \mathrm{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) : f(J^p) = 0 \text{ for some } p > 0\}$.

Lemma 12.5. *Let $J = \mathrm{Ker} \varepsilon$ be the augmentation ideal of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$, $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) = \{f \in \mathrm{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) : f(J^p) = 0 \text{ for some } p > 0\}$, where W is a free \mathcal{A}' -module. Equip $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ with the right action of $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ induced by multiplication on $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ from the left. Then the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ is injective.*

Proof. First observe that the algebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ is Noetherian and ideal J satisfies the so-called weak Artin–Rees property, i.e. for every finitely generated left $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module M and its submodule N there exists an integer $n > 0$ such that $J^n M \cap N \subset JN$. Indeed, observe that the algebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ can be equipped with a filtration similar to that introduced in Section 6 on the algebra $U_q^s(\mathfrak{g})$ in such a way that the associated graded algebra is finitely generated and semi-commutative (see (6.5)). The fact that $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ is Noetherian follows from the existence of the filtration on it for which the associated graded algebra is semi-commutative and from Theorem 4 in Ch. 5, §3 in [41] (compare also with Theorem 4.8 in [43]). The ideal J satisfies the weak Artin–Rees property because the subring $U_{\mathcal{A}'}^s(\mathfrak{m}_+) + Jt + J^2t^2 + \dots \subset U_{\mathcal{A}'}^s(\mathfrak{m}_+)[t]$, where t is a central indeterminate, is Noetherian (see [44], Ch. 11, §2, Lemma 2.1). The last fact follows from the existence of a filtration on $U_{\mathcal{A}'}^s(\mathfrak{m}_+) + Jt + J^2t^2 + \dots$ induced by the filtration on $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ for which the associated graded algebra is semi-commutative and again from Theorem 4 in Ch. 5, §3 in [41].

Finally, the module $\mathrm{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W)$ is obviously injective. By Lemma 3.2 in Ch. 3, [40] the module $\mathrm{hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) = \{f \in \mathrm{Hom}_{\mathcal{A}'}(U_{\mathcal{A}'}^s(\mathfrak{m}_+), W) : f(J^p) = 0 \text{ for some } p > 0\}$ is also injective since the ideal J satisfies the weak Artin–Rees property. \square

Lemma 12.5 implies that the complex $\mathrm{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$ is an injective resolution of $Q_{\mathcal{A}'}$ as a right $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module.

Now consider the complex

$$X_{\mathcal{A}'}^\bullet = \mathrm{Hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathbb{C}_{\mathcal{A}'}, \mathrm{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'}))$$

for calculating the functor

$$\mathrm{Ext}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_{\mathcal{A}'}, Q_{\mathcal{A}'}) = H^\bullet(X_{\mathcal{A}'}^\bullet).$$

Observe that the specialization of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $\mathbb{C}_{\mathcal{A}'}$ at ε is isomorphic to \mathbb{C}_ε , and the specialization of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $Q_{\mathcal{A}'}$ at ε is isomorphic to Q_ε . Therefore the specialization of the complex $X_{\mathcal{A}'}^\bullet$ at ε is isomorphic to

$$X_\varepsilon^\bullet = \mathrm{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, \mathrm{hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_\varepsilon^s(\mathfrak{m}_+)), Q_\varepsilon)),$$

where the complex

$$\mathrm{hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_\varepsilon^s(\mathfrak{m}_+)), Q_\varepsilon)$$

is defined similarly to $\mathrm{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\mathrm{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$ using the ε -specialization of the filtration $(U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k$. Applying the same arguments as in case of the complex $X_{\mathcal{A}'}^\bullet$, one can show that X_ε^\bullet is a complex for calculating the functor $\mathrm{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_\varepsilon, Q_\varepsilon) = H^\bullet(X_\varepsilon^\bullet)$.

The specialization of the algebra $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ at $q = 1$ is isomorphic to $U(\mathfrak{m}_+)$, the specialization of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $\mathbb{C}_{\mathcal{A}'}$ at $q = 1$ is isomorphic to the trivial representation \mathbb{C}_0 of $U(\mathfrak{m}_+)$, and the

specialization of the $U_{\mathcal{A}'}^s(\mathfrak{m}_+)$ -module $Q_{\mathcal{A}'}$ at $q = 1$ is isomorphic to $\mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}]$. By Theorem 11.3

$$\mathbb{C}[\overline{q(\mu_{M_+}^{-1}(u))}] \cong \mathbb{C}[M_+] \otimes W^s(G).$$

From Proposition 10.2 we obtain that the induced action of $U(\mathfrak{m}_+)$ on the corresponding variety $\overline{q(\mu_{M_+}^{-1}(u))}$ is obtained from the conjugation action of M_+ and now using proposition 11.2 one immediately deduces that the induced action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+] \otimes W^s(G)$ is generated by the action of $U(\mathfrak{m}_+)$ on $\mathbb{C}[M_+]$ by left invariant differential operators. Therefore the specialization of the complex $X_{\mathcal{A}'}^\bullet$ at $q = 1$ is isomorphic to

$$X_1^\bullet = \text{Hom}_{U(\mathfrak{m}_+)}(\mathbb{C}_0, \text{hom}_{U(\mathfrak{m}_+)}(\text{Bar}^\bullet(U(\mathfrak{m}_+)), \mathbb{C}[M_+] \otimes W^s(G))),$$

where the complex

$$\text{hom}_{U(\mathfrak{m}_+)}(\text{Bar}^\bullet(U(\mathfrak{m}_+)), \mathbb{C}[M_+] \otimes W^s(G))$$

is defined similarly to

$$\text{hom}_{U_{\mathcal{A}'}^s(\mathfrak{m}_+)}(\text{Bar}^\bullet(U_{\mathcal{A}'}^s(\mathfrak{m}_+)), Q_{\mathcal{A}'})$$

using the $q = 1$ -specialization of the filtration $(U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k$. Applying the same arguments as in case of the complex $X_{\mathcal{A}'}^\bullet$ one can show that X_1^\bullet is a complex for calculating the functor

$$\text{Ext}_{U(\mathfrak{m}_+)}^\bullet(\mathbb{C}_0, \mathbb{C}[M_+] \otimes W^s(G)) = H^\bullet(X_1^\bullet).$$

We also obviously have $\text{Ext}_{U(\mathfrak{m}_+)}^\bullet(\mathbb{C}_0, \mathbb{C}[M_+] \otimes W^s(G)) = \text{Ext}_{U(\mathfrak{m}_+)}^\bullet(\mathbb{C}_0, \mathbb{C}[M_+] \otimes W^s(G)) = H_{dR}^\bullet(M_+) \otimes W^s(G)$, where $H_{dR}^\bullet(M_+)$ is the de Rham cohomology of the unipotent group M_+ . Since $H_{dR}^n(M_+) = 0$ for $n > 0$ we deduce that $H^n(X_1^\bullet) = 0$ for $n > 0$.

Finally observe that the complex $X_{\mathcal{A}'}^\bullet$ and its specializations introduced above can be equipped with compatible filtrations by finitely generated free subcomplexes. These filtrations are induced by the filtrations $(Q_{\mathcal{A}'})_k$ and $(U_{\mathcal{A}'}^s(\mathfrak{m}_+))_k$ and by their specializations at $q = \varepsilon$ and $q = 1$. The Grothendieck cohomology semicontinuity property holds for these subcomplexes, and hence for the complex $X_{\mathcal{A}'}^\bullet$ as well. Therefore from the vanishing property $H^n(X_1^\bullet) = 0$ for $n > 0$ we deduce that for generic ε $\text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^n(\mathbb{C}_\varepsilon, Q_\varepsilon) = H^n(X_\varepsilon^\bullet) = 0$ for $n > 0$.

Now we prove that σ_ε is surjective. We start with the following lemma.

Lemma 12.6. *Let $\phi : X \rightarrow Y$ be an injective homomorphism of $U_\varepsilon^s(\mathfrak{m}_+)$ -modules. Assume that ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y , and that $\text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^1(\mathbb{C}_\varepsilon, X) = 0$, where \mathbb{C}_ε is the trivial representation of $U_\varepsilon^s(\mathfrak{m}_+)$. Suppose also that the action of the augmentation ideal J of $U_\varepsilon^s(\mathfrak{m}_+)$ on the cokernel of ϕ is locally nilpotent. Then ϕ is surjective.*

Proof. Consider the exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow W' \rightarrow 0,$$

where W' is the cokernel of ϕ , and the corresponding long exact sequence of cohomology,

$$\begin{aligned} 0 \rightarrow \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^0(\mathbb{C}_\varepsilon, X) \rightarrow \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^0(\mathbb{C}_\varepsilon, Y) \rightarrow \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^0(\mathbb{C}_\varepsilon, W') \rightarrow \\ \rightarrow \text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^1(\mathbb{C}_\varepsilon, X) \rightarrow \dots \end{aligned}$$

Since ϕ induces an isomorphism of the spaces of Whittaker vectors of X and of Y , and $\text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^1(\mathbb{C}_\varepsilon, X) = 0$, the initial part of the long exact cohomology sequence takes the form

$$0 \rightarrow \text{Wh}(X) \rightarrow \text{Wh}(Y) \rightarrow \text{Wh}(W') \rightarrow 0,$$

where the second map in the last sequence is an isomorphism. Using the last exact sequence we deduce that $\text{Wh}(W') = 0$. But the augmentation ideal J acts on W' by locally nilpotent transformations. Therefore, by Engel theorem, if W' is not trivial there should exist a nonzero $U_\varepsilon^s(\mathfrak{m}_+)$ -invariant vector in it. Thus we arrive at a contradiction, and $W' = 0$. Therefore ϕ is surjective.

□

Now recall that by (12.5) and (12.6) we already know that

$$\mathrm{Wh}(Q_\varepsilon) = W_\varepsilon^s(G), \quad \mathrm{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^1(\mathbb{C}_\varepsilon, Q_\varepsilon) = 0,$$

and by the definition of the module $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$

$$\mathrm{Wh}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))) = \mathrm{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))) = W_\varepsilon^s(G).$$

Observe also that by construction the map $\sigma_\varepsilon : Q_\varepsilon \rightarrow \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ induces an isomorphism of the spaces of Whittaker vectors. Since the action of the augmentation ideal J on $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ is locally nilpotent its action on the cokernel of σ_ε is locally nilpotent as well. Therefore σ_ε is surjective by Lemma 12.6.

Thus we have proved that $\sigma_\varepsilon : Q_\varepsilon \rightarrow \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ is an isomorphism of right $U_\varepsilon^s(\mathfrak{m}_+)$ -modules. Note that by the definitions of the spaces $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ and $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})$ we also have an obvious right $U_\varepsilon^s(\mathfrak{m}_+)$ -module isomorphism $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G)) = \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes W_\varepsilon^s(G)$.

Now consider the $U_\varepsilon^s(\mathfrak{m}_+)$ -submodule $\sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}))$ of Q_ε , where $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \subset \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$. Obviously $\sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \simeq \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})$ as a right $U_\varepsilon^s(\mathfrak{m}_+)$ -module.

Let $\phi_\varepsilon : \sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \otimes W_\varepsilon^s(G) \rightarrow Q_\varepsilon$ be the map induced by the action of $W_\varepsilon^s(G)$ on Q_ε . Since this action commutes with the adjoint action of $U_\varepsilon^s(\mathfrak{m}_+)$ on Q_ε we infer that ϕ_ε is a homomorphism of $U_\varepsilon^s(\mathfrak{m}_+)$ - $W_\varepsilon^s(G)$ -bimodules.

We claim that ϕ_ε is injective. This follows straightforwardly from lemma 12.4 because all Whittaker vectors of $\sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \otimes W_\varepsilon^s(G)$ belong to the subspace

$$1 \otimes W_\varepsilon^s(G) \subset \sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \otimes W_\varepsilon^s(G),$$

and the restriction of ϕ_ε to this subspace is injective.

Now we show that ϕ_ε is surjective. By the specializing the result of Lemma 12.5 at $q = \varepsilon$ one can immediately deduce that the right $U_\varepsilon^s(\mathfrak{m}_+)$ -module $\sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \otimes W_\varepsilon^s(G) \simeq \mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), W_\varepsilon^s(G))$ is injective. In particular, $\mathrm{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^1(\mathbb{C}_\varepsilon, \sigma_\varepsilon^{-1}(\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C})) \otimes W_\varepsilon^s(G)) = 0$. One checks straightforwardly, similarly the case of the map σ_ε , that the other conditions of Lemma 12.6 for the map ϕ_ε are satisfied as well. Therefore ϕ_ε is surjective.

This completes the proof of the proposition. □

Now we formulate our main statement.

Theorem 12.7. *Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Then for generic $\varepsilon \in \mathbb{C}$ the functor $E \mapsto Q_\varepsilon \otimes_{W_\varepsilon^s(G)} E$, is an equivalence of the category of finitely generated left $W_\varepsilon^s(G)$ -modules and the category $\mathbb{C}_\varepsilon[G_*] - \mathrm{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{X_\varepsilon^s}$. The inverse equivalence is given by the functor $V \mapsto \mathrm{Wh}(V)$. In particular, the latter functor is exact.*

Every module $V \in \mathbb{C}_\varepsilon[G_] - \mathrm{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{X_\varepsilon^s}$ is isomorphic to $\mathrm{hom}_\mathbb{C}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes \mathrm{Wh}(V)$ as a right $U_\varepsilon^s(\mathfrak{m}_+)$ -module. In particular, V is $U_\varepsilon^s(\mathfrak{m}_+)$ -injective, and $\mathrm{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_\varepsilon, V) = \mathrm{Wh}(V)$.*

Proof. Let E be a finitely generated $W_\varepsilon^s(G)$ -module. First we observe that by the definition of the algebra W_ε^s we have $\mathrm{Wh}(Q_{A'} \otimes_{W_\varepsilon^s(G)} E) = E$. Therefore to prove the theorem it suffices to check that for any $V \in \mathbb{C}_\varepsilon[G_*] - \mathrm{mod}_{U_\varepsilon^s(\mathfrak{m}_+)_{loc}}^{X_\varepsilon^s}$ the canonical map $f : Q_\varepsilon \otimes_{W_\varepsilon^s(G)} \mathrm{Wh}(V) \rightarrow V$ is an isomorphism.

By the previous Proposition $Q_\varepsilon = \text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes W_\varepsilon^s(G)$ as a $U_\varepsilon^s(\mathfrak{m}_+)$ – $W_\varepsilon^s(G)$ –bimodule. Therefore

$$(12.9) \quad Q_\varepsilon \otimes_{W_\varepsilon^s(G)} \text{Wh}(V) = \text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes \text{Wh}(V)$$

as a right $U_\varepsilon^s(\mathfrak{m}_+)$ –module.

Now the fact that f is an isomorphism can be established by repeating verbatim the arguments used in the proof of a similar statement for the map ϕ_ε in the previous Proposition. In particular f is injective by Lemma 12.4, $Q_\varepsilon \otimes_{W_\varepsilon^s(G)} \text{Wh}(V) = \text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes \text{Wh}(V)$ is an injective right $U_\varepsilon^s(\mathfrak{m}_+)$ –module by Lemma 12.5, and f is surjective by Lemma 12.6.

This completes the proof of the theorem. \square

13. LOCALIZATION OF QUANTUM BIEQUIVARIANT \mathcal{D} -MODULES

In this section we present a biequivariant version of the localization theorem for quantum \mathcal{D} –modules proved in [1, 33]. A similar result for Beilinson–Bernstein localization of \mathcal{D} modules on the flag variety was already mentioned in the original paper [2] (see also [16] for more details).

Let $\varepsilon \in \mathbb{C}$ be transcendental and generic. Denote by $\mathbb{C}_\varepsilon[G]$ the Hopf algebra generated by matrix coefficients of finite–dimensional representations of $U_\varepsilon^s(\mathfrak{g})$. There is a natural paring $(\cdot, \cdot) : U_\varepsilon^s(\mathfrak{g}) \otimes \mathbb{C}_\varepsilon[G] \rightarrow \mathbb{C}$. The algebra $\mathbb{C}_\varepsilon[G]$ is equipped with a $U_\varepsilon^s(\mathfrak{g})$ –bimodule structure via the left and the right regular action,

$$(13.1) \quad u(a) = a_1(u, a_2), \quad (a)u = (u, a_1)a_2, \quad u \in U_\varepsilon^s(\mathfrak{g}), \quad a \in \mathbb{C}_\varepsilon[G], \quad \Delta a = a_1 \otimes a_2.$$

Let \mathcal{D}_ε be the Heisenberg double of $U_\varepsilon^s(\mathfrak{g})$ defined in [26]. As a vector space $\mathcal{D}_\varepsilon = \mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})$, and the multiplication on \mathcal{D}_ε is given by

$$(13.2) \quad a \otimes u \cdot b \otimes v = au_1(b) \otimes u_2v, \quad a, b \in \mathbb{C}_\varepsilon[G], \quad u, v \in U_\varepsilon^s(\mathfrak{g}), \quad \Delta_s u = u_1 \otimes u_2.$$

The Heisenberg double is an analogue of the algebra of differential operators on the group G in case of Hopf algebras. \mathcal{D}_ε also has the structure of a $U_\varepsilon^s(\mathfrak{g})$ –bimodule,

$$(13.3) \quad \begin{aligned} u_L(a \otimes v) &= u_{(1)}(a) \otimes u_{(2)}vS_s(u_{(3)}), \quad u_R(a \otimes v) = (a)u \otimes v, \\ u &\in U_\varepsilon^s(\mathfrak{g}), \quad a \in \mathbb{C}_\varepsilon[G], \quad (id \otimes \Delta_s)\Delta_s(u) = u_{(1)} \otimes u_{(2)} \otimes u_{(3)}. \end{aligned}$$

Both the left and the right $U_\varepsilon^s(\mathfrak{g})$ actions on \mathcal{D}_ε are derivations with respect to the multiplicative structure in the sense that

$$(13.4) \quad u_L(a \otimes u \cdot b \otimes v) = u_{1L}(a \otimes u) \cdot u_{2L}(b \otimes v), \quad u_R(a \otimes u \cdot b \otimes v) = u_{1R}(a \otimes u) \cdot u_{2R}(b \otimes v).$$

These actions are analogues of the actions generated by left and right translations on G on the algebra of differential operators.

Let λ be a character of $U_\varepsilon^s(\mathfrak{h})$. λ naturally extends to a one–dimensional $U_\varepsilon^s(\mathfrak{b}_+)$ –module that we denote by \mathbb{C}_λ .

Note that there is an algebra embedding $U_\varepsilon^s(\mathfrak{g}) \subset \mathcal{D}_\varepsilon$, $x \mapsto 1 \otimes x$. The image of this embedding is an analogue of the algebra of right invariant vector fields on G . As in case of Lie groups right invariant vector fields generate left translations in the sense that

$$1 \otimes y_1 \cdot a \otimes x \cdot 1 \otimes S_s y_2 = y_L(a \otimes x), \quad y \in U_\varepsilon^s(\mathfrak{g}) \subset \mathcal{D}_\varepsilon, \quad a \otimes x \in \mathcal{D}_\varepsilon, \quad \Delta_s(y) = y_1 \otimes y_2.$$

Let $\mathbb{C}_\varepsilon[B_+]'$ be the quotient Hopf algebra of $\mathbb{C}_\varepsilon[G]$ by the Hopf algebra ideal generated by elements vanishing on $U_\varepsilon^s(\mathfrak{b}_+)$. Note that if V is a right $\mathbb{C}_\varepsilon[B_+]'$ –comodule then V is also naturally a left $U_\varepsilon^s(\mathfrak{b}_+)$ –module.

A $(U_\varepsilon^s(\mathfrak{b}_+), \lambda)$ –equivariant \mathcal{D}_ε –module is a triple (M, α, β) , where M is a complex vector space equipped with a left \mathcal{D}_ε –action $\alpha : \mathcal{D}_\varepsilon \times M \rightarrow M$, a right $\mathbb{C}_\varepsilon[B_+]'$ –coaction which gives rise to a left $U_\varepsilon^s(\mathfrak{b}_+)$ –action $\beta : U_\varepsilon^s(\mathfrak{b}_+) \times M \rightarrow M$ such that

- (1) The $U_\varepsilon^s(\mathfrak{b}_+)$ -actions on $M \otimes \mathbb{C}_\lambda$ given by $\beta \otimes \lambda$ and by $\alpha|_{U_\varepsilon^s(\mathfrak{b}_+)} \otimes \text{Id}$ coincide;
- (2) $\beta(u)(\alpha(a \otimes v)m) = \alpha(u_{1L}(a \otimes v))\beta(u_2)m$, for all $u \in U_\varepsilon^s(\mathfrak{b}_+)$, $a \otimes v \in \mathcal{D}_\varepsilon$, $m \in M$, $\Delta_s u = u_1 \otimes u_2$.

$(U_\varepsilon^s(\mathfrak{b}_+), \lambda)$ -equivariant \mathcal{D}_ε -modules form a category $\mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ morphisms in which are linear maps of vector spaces respecting all the above introduced structures on $(U_\varepsilon^s(\mathfrak{b}_+), \lambda)$ -equivariant \mathcal{D}_ε -modules.

Let $\mathcal{D}_\varepsilon^\lambda$ be the maximal quotient of \mathcal{D}_ε which is an object of $\mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$. In fact one has

$$\mathcal{D}_\varepsilon^\lambda \simeq \mathcal{D}_\varepsilon / \mathcal{D}_\varepsilon I,$$

where I is the left ideal in \mathcal{D}_ε generated by the elements $1 \otimes e_i$, $1 \otimes t_i - \lambda(t_i)$, $i = 1, \dots, l$. We denote by $\mathbf{1}$ the image of $1 \otimes 1 \in \mathcal{D}_\varepsilon$ in $\mathcal{D}_\varepsilon^\lambda$.

Now define the global section functor $\Gamma : \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda \rightarrow \text{Vect}_{\mathbb{C}}$, where $\text{Vect}_{\mathbb{C}}$ is the category of vector spaces,

$$\Gamma(M) = \text{Hom}_{\mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda}(\mathcal{D}_\varepsilon^\lambda, M) = \text{Hom}_{U_\varepsilon^s(\mathfrak{b}_+)}(\mathbb{C}_\varepsilon, M),$$

where in the last formula $U_\varepsilon^s(\mathfrak{b}_+)$ acts on M according to β -action, and \mathbb{C}_ε is the trivial representation of $U_\varepsilon^s(\mathfrak{b}_+)$ given by the counit.

One can also write

$$(13.5) \quad \Gamma(M) = \text{Hom}_{U_\varepsilon^s(\mathfrak{b}_+)}(\mathbb{C}_\lambda, M),$$

where $U_\varepsilon^s(\mathfrak{b}_+)$ acts on M according to the α -action composed with the embedding $U_\varepsilon^s(\mathfrak{b}_+) \rightarrow \mathcal{D}_\varepsilon$, $x \mapsto 1 \otimes x$.

Naturally $\Gamma(\mathcal{D}_\varepsilon^\lambda) = \text{End}_{\mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda}(\mathcal{D}_\varepsilon^\lambda)$ is an algebra with multiplication induced from \mathcal{D}_ε . The algebra $\Gamma(\mathcal{D}_\varepsilon^\lambda)$ naturally acts from the left on spaces $\Gamma(M)$ for $M \in \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$.

Recall that there is a locally finite right adjoint action of $\text{Ad} : U_\varepsilon^s(\mathfrak{g}) \times U_\varepsilon^s(\mathfrak{g})^{fin} \rightarrow U_\varepsilon^s(\mathfrak{g})^{fin}$ given by

$$\text{Ad}x(w) = S_s^{-1}(x_2)wx_1,$$

where $\Delta_s(x) = x_1 \otimes x_2$, $x \in U_\varepsilon^s(\mathfrak{g})$, $w \in U_\varepsilon^s(\mathfrak{g})^{fin}$. Let $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g})^{fin} \rightarrow \mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})^{fin}$ be the dual $\mathbb{C}_\varepsilon[G]$ -coaction on $U_\varepsilon^s(\mathfrak{g})^{fin}$. One can consider the tensor product $\mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})^{fin}$ as a linear subspace of \mathcal{D}_ε . Using this fact Δ_{Ad} can be regarded as a linear map to \mathcal{D}_ε , $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g})^{fin} \rightarrow \mathcal{D}_\varepsilon$. In fact Δ_{Ad} is an embedding, the left inverse map $\Delta_{\text{Ad}S_s}$ is given by

$$(13.6) \quad \Delta_{\text{Ad}S_s}(a \otimes x) = a \otimes 1 \cdot \Delta_{\text{Ad}S_s}(x), \quad a \otimes x \in \text{Im} \Delta_{\text{Ad}} \subset \mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})^{fin},$$

where $\Delta_{\text{Ad}S_s} : U_\varepsilon^s(\mathfrak{g})^{fin} \rightarrow \mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})^{fin}$ is the map dual to the action of $U_\varepsilon^s(\mathfrak{g})$ on $U_\varepsilon^s(\mathfrak{g})^{fin}$ given by $\text{Ad}S_s$, and the image of $\Delta_{\text{Ad}S_s}$ in (13.6) belongs to the subspace $1 \otimes U_\varepsilon^s(\mathfrak{g})^{fin}$ which is naturally identified with $U_\varepsilon^s(\mathfrak{g})^{fin}$.

Direct calculation also shows that $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g})^{fin} \rightarrow \mathcal{D}_\varepsilon$ is an algebra antihomomorphism,

$$\Delta_{\text{Ad}}(x) \cdot \Delta_{\text{Ad}}(y) = \Delta_{\text{Ad}}(yx).$$

Note that Δ_{Ad} can be extended to a homomorphism from $U_\varepsilon^s(\mathfrak{g})$ to a certain completion $\mathbb{C}_\varepsilon[G] \widehat{\otimes} U_\varepsilon^s(\mathfrak{g})$ of $\mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})$ by infinite series terms of which are elements of $\mathbb{C}_\varepsilon[G] \otimes U_\varepsilon^s(\mathfrak{g})$. We denote this extension by the same symbol, $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g}) \rightarrow \mathbb{C}_\varepsilon[G] \widehat{\otimes} U_\varepsilon^s(\mathfrak{g})$. One can equip the completion $\mathbb{C}_\varepsilon[G] \widehat{\otimes} U_\varepsilon^s(\mathfrak{g})$ with a multiplication induced from \mathcal{D}_ε . We denote the obtained algebra by $\widehat{\mathcal{D}}_\varepsilon$.

One checks that the map $\Delta_{\text{Ad}S_s}$ naturally extends to a left inverse of $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g}) \rightarrow \widehat{\mathcal{D}}_\varepsilon$, and hence $\Delta_{\text{Ad}} : U_\varepsilon^s(\mathfrak{g}) \rightarrow \widehat{\mathcal{D}}_\varepsilon$ is an embedding. The image of this map can be regarded as an analogue of the algebra of left invariant vector fields on G . In particular, these analogues generate the right action of $U_\varepsilon^s(\mathfrak{g})$ on \mathcal{D}_ε ,

$$(13.7) \quad \Delta_{\text{Ad}}(y_1) \cdot a \otimes x \cdot \Delta_{\text{Ad}}(S_s^{-1}y_2) = y_R(a \otimes x), \quad y \in U_\varepsilon^s(\mathfrak{g}), \quad \Delta_s(y) = y_1 \otimes y_2, \quad a \otimes x \in \mathcal{D}_\varepsilon.$$

The map Δ_{Ad} is also equivariant with respect to the right action of $U_\varepsilon^s(\mathfrak{g})$ on \mathcal{D}_ε in the sense that

$$(13.8) \quad u_R(\Delta_{\text{Ad}}(v)) = \Delta_{\text{Ad}}(\text{Adu}(v)), \quad u \in U_\varepsilon^s(\mathfrak{g}), \quad v \in U_\varepsilon^s(\mathfrak{g})^{\text{fin}}.$$

Denote by J_λ the annihilator of the Verma module $M_\varepsilon(\lambda) = U_\varepsilon^s(\mathfrak{g}) \otimes_{U_\varepsilon^s(\mathfrak{b}_+)} \mathbb{C}_\lambda$ in $U_\varepsilon^s(\mathfrak{g})^{\text{fin}}$. J_λ is generated by the ideal of the center $Z(U_\varepsilon^s(\mathfrak{g})^{\text{fin}}) = Z(U_\varepsilon^s(\mathfrak{g}))$ corresponding to a character $\chi_{\lambda+\rho} : Z(U_\varepsilon^s(\mathfrak{g})) \rightarrow \mathbb{C}$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in P_+$. Let $U_\varepsilon^s(\mathfrak{g})^\lambda$ be the quotient of $U_\varepsilon^s(\mathfrak{g})^{\text{fin}}$ by J_λ , $U_\varepsilon^s(\mathfrak{g})^\lambda = U_\varepsilon^s(\mathfrak{g})^{\text{fin}} / J_\lambda$.

We also denote by I_λ the annihilator of $M_\varepsilon(\lambda)$ in $U_\varepsilon^s(\mathfrak{g})$ and by $U_\varepsilon^s(\mathfrak{g})_\lambda$ the quotient $U_\varepsilon^s(\mathfrak{g})_\lambda = U_\varepsilon^s(\mathfrak{g}) / I_\lambda$.

A character $\lambda : U_\varepsilon^s(\mathfrak{h}) \rightarrow \mathbb{C}$ is called regular dominant if for each $\phi \in P_+$ and all weights ψ of $V_\varepsilon(\phi)$, $\phi \neq \psi$, one has $\chi_{\lambda+\phi} \neq \chi_{\lambda+\psi}$.

Proposition 13.1. ([1], **Proposition 4.8, Theorem 4.12**) *The map*

$$(13.9) \quad U_\varepsilon^s(\mathfrak{g})^\lambda \rightarrow \Gamma(\mathcal{D}_\varepsilon^\lambda)^{\text{opp}} = \text{Hom}_{U_\varepsilon^s(\mathfrak{b}_+)}(\mathbb{C}_\varepsilon, \mathcal{D}_\varepsilon^\lambda), x \mapsto \Delta_{\text{Ad}}(x)\mathbf{1}$$

is an algebra isomorphism.

If λ is regular dominant the global section functor $\Gamma : \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda \rightarrow \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ is an equivalence of the category $\mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ and of the category $\text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ of right $U_\varepsilon^s(\mathfrak{g})^\lambda$ -modules. The inverse functor is given by

$$(13.10) \quad V \mapsto V \otimes_{U_\varepsilon^s(\mathfrak{g})^\lambda} \mathcal{D}_\varepsilon^\lambda, \quad V \in \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda.$$

Now we present an equivariant version of the previous proposition. Let $U \subset U_\varepsilon^s(\mathfrak{g})$ be a subalgebra equipped with a character $\chi : U \rightarrow \mathbb{C}$. Denote by \mathbb{C}_χ the corresponding one-dimensional representation of U . Assume that U is also a coideal, i.e. $\Delta_s(U) \subset U \otimes U_\varepsilon^s(\mathfrak{g})$.

A biequivariant \mathcal{D}_ε -module is a $(U_\varepsilon^s(\mathfrak{b}_+), \lambda)$ -equivariant \mathcal{D}_ε -module M which is also equipped with the structure of a left U -module $\gamma : U \times M \rightarrow M$ such that

- (1) For any $u \in U$ the action of the operator $\chi(u_1)\alpha(\Delta_{\text{Ad}}(S_s u_2))$ on M is well defined, and the U -actions on $\mathbb{C}_\chi \otimes M$ given by $\text{Id} \otimes \gamma$ and by $\chi \otimes \alpha \Delta_{\text{Ad}} \circ S_s$ coincide;
- (2) $\gamma(u)(\alpha(a \otimes v)m) = \alpha(S_s(u_2)_R(a \otimes v))\gamma(u_1)m$, for all $u \in U$, $a \otimes v \in \mathcal{D}_\varepsilon$, $m \in M$, $\Delta_s u = u_1 \otimes u_2$.

Biequivariant \mathcal{D}_ε -modules form a category ${}^\chi_U \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ morphisms in which are linear maps of vector spaces respecting all the above introduced structures on biequivariant \mathcal{D}_ε -modules.

A (U, χ) -equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -module is a right $U_\varepsilon^s(\mathfrak{g})^\lambda$ -module V equipped with the structure of a left U -module $\gamma : U \times V \rightarrow V$ such that

- (1) For any $u \in U$ and $v \in V$ one has $\gamma(u)m = \chi(u_1)S_s u_2 v$, where a priori $\chi(u_1)S_s u_2 m$ should be understood as the natural action of the element $\chi(u_1)S_s u_2$ on the induced $U_\varepsilon^s(\mathfrak{g})$ -module $V' = V \otimes_{U_\varepsilon^s(\mathfrak{g})^\lambda} U_\varepsilon^s(\mathfrak{g})$;
- (2) $\gamma(u)(xv) = \text{Ad} S_s(u_2)(x)(\gamma(u_1)v)$, for all $u \in U$, $x \in U_\varepsilon^s(\mathfrak{g})^\lambda$, $v \in V$, $\Delta_s u = u_1 \otimes u_2$.

(U, χ) -equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -modules form a category ${}^\chi_U \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ morphisms in which are linear maps of vector spaces respecting all the above introduced structures on equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -modules.

Formula (13.5), condition (2) in the definition of biequivariant \mathcal{D}_ε -modules and the obvious relation

$$u_R(1 \otimes x) = \varepsilon(u)1 \otimes x, \quad u, x \in U_\varepsilon^s(\mathfrak{g})$$

imply that if M is a biequivariant \mathcal{D}_ε -module then γ induces a U -action on $\Gamma(M)$. From formula (13.8) it also follows that if M is a biequivariant \mathcal{D}_ε -module then $\Gamma(M)$ is an equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -module. Conversely, the second relation in (13.4) and (13.7) imply that if V is an equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -module with an equivariant structure γ then the formula

$$(13.11) \quad \gamma(u)(v \otimes (a \otimes x)) = \gamma(u_1)(v) \otimes S_s(u_2)_R(a \otimes x), \quad v \in V, \quad a \otimes x \in \mathcal{D}_\varepsilon^\lambda, \quad u \in U$$

defines the structure of a biequivariant \mathcal{D}_ε -module on $V \otimes_{U_\varepsilon(\mathfrak{g})^\lambda} \mathcal{D}_\varepsilon^\lambda$.

Thus we have the following proposition.

Proposition 13.2. *If λ is regular dominant the global section functor $\Gamma : \mathcal{D}_{U_\varepsilon(\mathfrak{b}_+)}^\lambda \rightarrow \text{mod} - U_\varepsilon(\mathfrak{g})^\lambda$ gives rise to an equivalence of the category ${}^\chi_U \mathcal{D}_{U_\varepsilon(\mathfrak{b}_+)}^\lambda$ of biequivariant \mathcal{D}_ε -modules and of the category ${}^\chi_U \text{mod} - U_\varepsilon(\mathfrak{g})^\lambda$ of equivariant right $U_\varepsilon(\mathfrak{g})^\lambda$ -modules. The inverse functor is given by*

$$(13.12) \quad V \mapsto V \otimes_{U_\varepsilon(\mathfrak{g})^\lambda} \mathcal{D}_\varepsilon^\lambda, \quad V \in {}^\chi_U \text{mod} - U_\varepsilon(\mathfrak{g})^\lambda.$$

Denote by I_ε^r the right ideal in $\mathbb{C}_\varepsilon[G^*]$ generated by the kernel of χ_ε^s in $\mathbb{C}_\varepsilon[M_-]$, and by $\rho_{\chi_\varepsilon^s}$ the canonical projection $\mathbb{C}_\varepsilon[G^*] \rightarrow I_\varepsilon^r \backslash \mathbb{C}_\varepsilon[G^*]$. Let Q_ε^r be the image of $\mathbb{C}_\varepsilon[G^*]$ under the projection $\rho_{\chi_\varepsilon^s}$.

Assume that the roots $\gamma_1, \dots, \gamma_n$ are simple or that the set $\gamma_1, \dots, \gamma_n$ is empty, and hence the segment Δ_{s^1} is of the form $\Delta_{s^1} = \{\gamma_1, \dots, \gamma_n\}$. Then from formula (4.13) it follows that $\Delta_s(U_\varepsilon^s(\mathfrak{m}_+)) \subset U_\varepsilon^s(\mathfrak{m}_+) \otimes U_\varepsilon^s(\mathfrak{b}_+)$.

Similarly to Section 9 we deduce that the left action $\text{Ad} \circ S_s$ of $U_\varepsilon^s(\mathfrak{m}_+)$ on $\mathbb{C}_\varepsilon[G^*]$ induces an action on Q_ε^r which we denote by $\text{Ad} \circ S_s$. One can also define the corresponding W -algebra by

$$W_\varepsilon^s(G)^r = \text{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, Q_\varepsilon^r),$$

where the multiplication in $W_\varepsilon^s(G)^r$ is induced from $\mathbb{C}_\varepsilon[G^*]$.

As in Proposition 11.4 we have an embedding

$$Z(\mathbb{C}_\varepsilon[G^*]) \rightarrow W_\varepsilon^s(G)^r.$$

Note that by Proposition 9.3 $\mathbb{C}_\varepsilon[G^*] \simeq U_\varepsilon^s(\mathfrak{g})^{fin}$. Let Z_λ be the kernel of the character $\chi_\lambda : Z(\mathbb{C}_\varepsilon[G^*]) \rightarrow \mathbb{C}$. Consider the quotient

$$W_\varepsilon^s(G)_\lambda^r = W_\varepsilon^s(G)^r / W_\varepsilon^s(G)^r Z_\lambda.$$

Observe that for generic ε we have an algebra isomorphism $\mathbb{C}_\varepsilon[M_-] = U_\varepsilon^s(\mathfrak{m}_+)$ and that $U_\varepsilon^s(\mathfrak{m}_+)$ is a coideal in $U_\varepsilon^s(\mathfrak{g})$. In particular, χ_ε^s is a character of $U_\varepsilon^s(\mathfrak{m}_+)$. Therefore one can define the category ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ of $(U_\varepsilon^s(\mathfrak{m}_+), \chi_\varepsilon^s)$ -equivariant $U_\varepsilon^s(\mathfrak{g})^\lambda$ -modules. Consider the full subcategory ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$ of ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ objects of which are finitely generated over $U_\varepsilon^s(\mathfrak{g})$ objects of ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})^\lambda$ such that the γ -action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_+)$ on them is locally nilpotent.

Let $Q_{\varepsilon, \lambda}^r$ be the image of $U_\varepsilon^s(\mathfrak{g})^\lambda$ under the projection $\rho_{\chi_\varepsilon^s}$. We have the following straightforward analogue of Theorem 12.7 for the category ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$.

Proposition 13.3. *Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_{l'}$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_{l'}$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Then for generic $\varepsilon \in \mathbb{C}$ the functor $E \mapsto E \otimes_{W_\varepsilon^s(G)_\lambda^r} Q_{\varepsilon, \lambda}^r$ is an equivalence of the category of finitely generated right $W_\varepsilon^s(G)_\lambda^r$ -modules and the category ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$. The inverse equivalence is given by the functor $V \mapsto \text{Hom}_{U_\varepsilon^s(\mathfrak{m}_+)}(\mathbb{C}_\varepsilon, V) = \text{Wh}(V)$. In particular, the latter functor is exact.*

Every module $V \in {}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$ is isomorphic to $\text{hom}_{\mathbb{C}}(U_\varepsilon^s(\mathfrak{m}_+), \mathbb{C}) \otimes \text{Wh}(V)$ as a left $U_\varepsilon^s(\mathfrak{m}_+)$ -module. In particular, V is $U_\varepsilon^s(\mathfrak{m}_+)$ -injective, and $\text{Ext}_{U_\varepsilon^s(\mathfrak{m}_+)}^\bullet(\mathbb{C}_\varepsilon, V) = \text{Wh}(V)$.

Let $\mathbb{C}_\varepsilon[M_+]'$ be the coalgebra which is the quotient of $\mathbb{C}_\varepsilon[G]$ by the coalgebra ideal generated by elements vanishing on $U_\varepsilon^s(\mathfrak{m}_+)$. Proposition 13.3 implies that the $U_\varepsilon^s(\mathfrak{m}_+)$ -action on the objects of the category ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} - U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$ is induced by the adjoint $U_\varepsilon^s(\mathfrak{g})$ -action on $U_\varepsilon^s(\mathfrak{g})^\lambda$ which is locally finite. Therefore this action gives rise to a right coaction of $\mathbb{C}_\varepsilon[M_+]'$ on objects of ${}_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \text{mod} -$

$U_\varepsilon^s(\mathfrak{g})_{loc}^\lambda$. Conversely, a right $\mathbb{C}_\varepsilon[M_+]'$ -coaction on any such object V gives rise to a $U_\varepsilon^s(\mathfrak{m}_+)$ -action such that the action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{m}_+)$ on it is locally nilpotent. Indeed, the action of the augmentation ideal of $U_\varepsilon^s(\mathfrak{b}_+)$ on any finite-dimensional $U_\varepsilon^s(\mathfrak{g})$ -module is locally nilpotent, and hence the action of $U_\varepsilon^s(\mathfrak{m}_+) \subset U_\varepsilon^s(\mathfrak{b}_+)$ induced by the coaction of $\mathbb{C}_\varepsilon[M_+]'$ is locally nilpotent as well.

Now observe that in this case the $U_\varepsilon^s(\mathfrak{m}_+)$ -action defined by formula (13.11) on the corresponding biequivariant \mathcal{D}_ε -module gives rise to a right $\mathbb{C}_\varepsilon[M_+]'$ -coaction which is the tensor product of the right coaction of $\mathbb{C}_\varepsilon[M_+]'$ on V described above and the right coaction of $\mathbb{C}_\varepsilon[M_+]'$ on $\mathcal{D}_\varepsilon^\lambda$ induced by the regular action $(u, a) \mapsto (a)S_s(u)$, $u \in U_\varepsilon^s(\mathfrak{g})$, $a \in \mathbb{C}_\varepsilon[G]$, of $U_\varepsilon^s(\mathfrak{g})$ on $\mathbb{C}_\varepsilon[G]$ which is locally finite by definition.

Conversely, if M is an object of the category $_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ such that the γ -action of $U_\varepsilon^s(\mathfrak{m}_+)$ on it is induced by a right $\mathbb{C}_\varepsilon[M_+]'$ -coaction then the induced $U_\varepsilon^s(\mathfrak{m}_+)$ -action on $\Gamma(M)$ corresponds to a right $\mathbb{C}_\varepsilon[M_+]'$ -coaction on $\Gamma(M)$.

Now consider the full subcategory $_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ of $_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ objects of which are finitely generated over \mathcal{D}_ε objects of $_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ such that for each $M \in _{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ the γ -action of $U_\varepsilon^s(\mathfrak{m}_+)$ on M is induced by a right $\mathbb{C}_\varepsilon[M_+]'$ -coaction. From Propositions 13.2 and 13.3 and the discussion above we immediately obtain the following statement.

Theorem 13.4. *Assume that the roots $\gamma_1, \dots, \gamma_n$ (or $\gamma_{n+1}, \dots, \gamma_\nu$) are simple or one of the sets $\gamma_1, \dots, \gamma_n$ or $\gamma_{n+1}, \dots, \gamma_\nu$ is empty. Suppose also that the numbers t_i defined in (10.5) are not equal to zero for all i . Suppose also that λ is regular dominant. Then for generic transcendental $\varepsilon \in \mathbb{C}$ the category $_{U_\varepsilon^s(\mathfrak{m}_+)}^{\chi_\varepsilon^s} \mathcal{D}_{U_\varepsilon^s(\mathfrak{b}_+)}^\lambda$ is equivalent to the category of finitely generated right $W_\varepsilon^s(G)_\lambda^r$ -modules.*

REFERENCES

- [1] Backelin, E., Kremnizer, K., Quantum flag varieties, equivariant quantum \mathcal{D} -modules, and localization of quantum groups, *Adv. Math.* **203** (2006), 408-429.
- [2] Beilinson, A., Bernstein, J., Localisation de \mathfrak{g} -modules, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 15-18.
- [3] Bourbaki, N., Groupes et algebras de Lie, Chap. 4,5,6, Paris, Hermann (1968).
- [4] Belavin, A.A., Drinfeld, V.G., Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funct. Anal. Appl.* **16** (1981), 159-180.
- [5] Carter, R., Simple groups of Lie type, John Wiley & Sons (1972).
- [6] Carter, R. W., Conjugacy classes in the Weyl group, *Compositio Math.* **25** (1972), 1-59.
- [7] Chari, V., Pressley, A., A guide to quantum groups, Cambridge Univ. Press (1994).
- [8] De Concini, C., Kac, V. G., Representations of quantum groups at roots of 1: reduction to the exceptional case, *Int. J. Mod. Phys. A* **7** em Suppl. 1A (1992), 141-149.
- [9] Dodd, C., Kremnizer, Y., A Localization Theorem for Finite W-algebras, arXiv:0911.2210.
- [10] Drinfeld, V.G., Quantum groups, Proc. Int. Congr. Math. Berkley, California, 1986, Amer. Math. Soc., Providence (1987), 718-820.
- [11] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L., Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1989), 178-206.
- [12] Gan, W. L., Ginzburg, V., Quantization of Slodowy slices, *Int. Math. Res. Not.* **5** (2002), 243-255.
- [13] Joseph, A., Quantum groups and their primitive ideals. *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3), **29**, Springer-Verlag, Berlin (1995).
- [14] Joseph, A., Letzter, G., Local finiteness of the adjoint action for quantized enveloping algebras, *J. Algebra* **153** (1992), 289-318.
- [15] Kac, V.G., Infinite dimensional Lie algebras, Cambridge Univ. Press (1990).
- [16] Kashiwara, M., Representation theory and \mathcal{D} -modules on flag varieties. Orbites unipotentes et représentations, III. *Astérisque* **173-174** (1989), 55-109.
- [17] Khoroshkin, S.M., Tolstoy, V.N., Universal R-matrix for quantized (super)algebras, *Comm. Math. Phys.* **141** (1991), 599-617.

- [18] Khoroshkin, S. M., Tolstoy, V. N., The Cartan-Weyl basis and the universal R-matrix for quantum Kac-Moody algebras and superalgebras. Quantum symmetries (Clausthal, 1991), 336–351, World Sci. Publ., River Edge, NJ (1993).
- [19] Levendorskii, S. Z., Soibelman, Ya. S., Some applications of the quantum Weyl groups, *J. Geom. Phys.* **7** (1990), 241–254.
- [20] Lu, J.H., Momentum mapping and reduction of Poisson actions. In: Symplectic geometry, groupoids and integrable systems, P.Dazord and A.Weinstein (eds.), 209–226, Springer-Verlag (1989).
- [21] Premet, A., Special transverse slices and their enveloping algebras. With an appendix by Serge Skryabin, *Adv. Math.* **170** (2002), 1–55.
- [22] Reshetikhin, N.Yu., Semenov-Tian-Shansky, M.A., Factorization problems for quantum groups, in: *Geometry and Physics, essays in honour of I.M.Gelfand, S.Gindikin and I.M.Singer* (eds.), North Holland, Amsterdam - London - New York (1991), 533–550.
- [23] Richardson, R.W., Conjugacy classes of involutions in Coxeter groups, *Bull. Austral. Math. Soc.* **26** (1982), 1–15.
- [24] Semenov-Tian-Shansky, M.A., What is a classical r -matrix, *Funct. Anal. Appl.* **17** (1983), 17–33.
- [25] Semenov-Tian-Shansky, M.A., Dressing transformations and Poisson–Lie group actions, *Publ. Math. RIMS* **21** (1985), 1237–1260.
- [26] Semenov-Tian-Shansky, M.A., Poisson Lie groups, quantum duality principle and the quantum double, *Contemporary Math.* **175** (1994), 219–248.
- [27] Sevostyanov, A., Regular nilpotent elements and quantum groups, *Comm. Math. Phys.* **204** (1999), 1.
- [28] Sevostyanov, A. The Whittaker model of the center of the quantum group and Hecke algebras. PhD thesis, Uppsala (1999).
- [29] Sevostyanov, A., Quantum deformations of Whittaker modules and the Toda lattice, *Duke Math. J.* **105** (2000), 211–238.
- [30] Sevostyanov, A., Towards Drinfeld-Sokolov reduction for quantum groups, *J. Geom. Phys.* **33** (2000), 235–256.
- [31] Sevostyanov, A., Algebraic group analogues of Slodowy slices and deformations of Poisson W-algebras, *Int. Math. Res. Not.* **2011** (2011), 1880–1925.
- [32] Sevostyanov, A., Conjugacy classes in Weyl groups and q -W algebras, *Adv. Math.* **228** (2011), 1315–1376.
- [33] Tanisaki, T., The Beilinson-Bernstein correspondence for quantized enveloping algebras, *em Math. Z.* **250** (2005), 299–361.
- [34] Tolstoy, V. N., Extremal projectors for quantized Kac-Moody superalgebras and some of their applications. Quantum symmetries (Clausthal, 1989), 118–125, *Lecture Notes in Physics* **370** (1990).
- [35] Weinstein, A., Local structure of Poisson manifolds, *J. Different. Geom.* **18** (1983), 523–558.
- [36] Zelobenko, D.P., Compact Lie groups and their representations, Amer. Math. Soc., Providence (1973).
- [37] Zelobenko, D.P., Extremal cocycles on Weyl groups, *Funk. Anal. i.ego Pril.* **21**, no. 3 (1987), 11–21.
- [38] Buchweitz, R.-O., Liu, S., Hochschild cohomology and representation-finite algebras, *Proc. London Math. Soc.* (3) **88** (2004), 355–380.
- [39] Cartan, H., Eilenberg, S., Homological Algebra, Princeton (1956).
- [40] Hartshorne, R., Algebraic geometry, Springer, New York (2006).
- [41] Jacobson, N., Lie algebras, Interscience Publishers, New York (1962).
- [42] McConnell, J.C., Stafford, J. T., Gelfand–Kirillov dimension and associated graded modules, *Journal of Algebra* **125** (1989), 197–214.
- [43] McConnell, J.C., Quantum groups, filtered rings and Gelfand–Kirillov dimension, In *Lecture Notes in Math* **1448**, Springer, Berlin (1991), 139–149.
- [44] Passman, D. S., The algebraic structure of group rings, Jonh Wiley & Sons, New York (1977).